THE PROBLEM OF OPTIMAL CONTROL IN THE RUN MODEL

N.I. OVSYANNIKOVA, A.A. POPOVA

The article is devoted to the search of optimal control in a model of a middle-distance running in order to minimize the time spent. Control is the force exerted by a sportsman per unit mass – or rather, the acceleration. The initial conditions and constraints on the control are prescribed. In the article optimal control problem is reduced to a fixed-time one by introducing the coefficient of compression of time, which is the second controlling factor.

Key words: model of competition in running, optimal control in the problem of fast-acting, the coefficient of compression of time.

1. THEORETICAL ANALYSIS

Running competition is one of the oldest sports, but many records set back in the 90's of the past century cannot be beaten by athletes so far. So the question arises: is the absolute world record just the limit of human capabilities or do we need to reconsider the approach to training of athletes? Let's consider the running competition as an optimal control problem with a view to minimizing the time of the race with a set distance.

The mathematical formulation of the problem is as follows:

$$J(u) = T \to \inf \tag{1}$$

$$\begin{cases} x'(t) = v(t) \\ v'(t) = u(t) - r(v) \\ E'(t) = -\dfrac{u(t)v(t)}{\eta(v)} + d(E) \end{cases} \tag{2}$$

$$0 \le t \le T$$

$$0 \le E(t) \le E_0 \tag{3}$$

$$x(0) = 0,\ v(0) = v_0,\ E(0) = E_0 \tag{4}$$

$$x(T) = D \tag{5}$$

$$0 \le u(t) \le U \tag{6}$$

where $t \in [0; T]$ – time, x(t) – coordinate, v(t) – speed (rate), D – a given distance, E(t) – supply of energy in muscles, $u(t)v(t)\eta^{-1}(v)$ – energy expenditure due to the mechanical work, $\eta^{-1}(v)$ characterizes the transfer efficiency (efficiency of transition) of the chemical energy into mechanical one, u(t) – control function (the force developed by the athlete on 1 kg of its weight (mass), or acceleration), U – maximum value of acceleration, $d(E) = \gamma(E_1 - E)$ – an increase in energy due to the removal of lactic acid by blood and transfer it to other muscles, where it is oxidized, $\gamma^{-1} = 900c$, $E_1 = 0{,}65E_0$, E_0 – initial energy, r(v) – resistance to the motion to 1 kg of runner's weight (mass), which depends on the speed in a following way [1]:

$$r(v)=\begin{cases}0{,}0037v^2, & v\le 6\ м/с\\ 0{,}6(v-6)+0{,}0037v^2, & v>6\ м/с\end{cases}$$

2. METHOD OF SOLUTION

The speed-in-action problem is formulated as follows: it is needed to find the control vector u(t), which transforms the system from one given point to another within the shortest time. Reduce the speed-in-action problem to the problem with fixed time and free right end. Change the independent variable, putting $t=\xi\tau,\ dt=\xi d\tau$, where new independent variable τ varies on a fixed interval $[0,T_0]$. Here $\xi\ge 0$ – scaling number, which can be considered as the control and with the help of which we will change the interval of integration; T_0 – selected allowable value of time. Include constraints (5) and (3) in the functional with the help of penalty terms and move on to the following problem sequence:

$$J(u)=\xi T_0+M_k[D-x(\xi T_0)]^2+N_k\xi\int_0^{T_0}\left((\max[-E(\xi\tau),0])^2+(\max[E(\xi\tau)-E_{0,}\ 0])^2\right)d\tau\to\inf \tag{8}$$

$$\begin{cases}x'(\tau)=\xi v(t),\\ v'(\tau)=\xi(u(t)-r(v)),\\ E'(\tau)=\xi(d(E)-\dfrac{u(t)v(t)}{\eta(v)}),\\ t'(\tau)=\xi\end{cases} \tag{9}$$

$$0\le\xi\le 1,\ 0\le u(\tau)\le U \tag{10}$$

$$t(0)=0,\quad x(0)=0,\quad v(0)=v_0,\quad E(0)=E_0 \tag{11}$$

Divide the segment $[0,T]$ into q equal parts by the points $t_i=\Delta t\cdot i,\quad i=\overline{0,q}$, and the segment $[0,T_0]$ – by the points $\tau_i=\Delta\tau\cdot i,\quad i=\overline{0,q}$, where $\Delta t=\xi\cdot\Delta\tau,\quad \xi\in R$ and move on to the discrete optimal control problem:

$$I=\xi T_0+M_k(D-x^q)^2+N_k\Delta\tau\cdot\xi\sum_{i=0}^{q-1}([\max(-E^i,0)]^2+[\max(E^i-E_0,0)]^2)\to\inf \tag{12}$$

$$\begin{cases}x^{i+1}=x^i+\Delta\tau\cdot\xi v^i\\ v^{i+1}=v^i+\Delta\tau\cdot\xi(u^i-r(v^i))\\ E^{i+1}=E^i+\Delta\tau\cdot\xi(d(E^i)-\dfrac{u^iv^i}{\eta(v^i)})\\ t^{i+1}=t^i+\Delta\tau\cdot\xi\\ i=\overline{0,q-1}\end{cases} \tag{13}$$

$$0\le u^i\le U,\quad i=\overline{0,q-1} \tag{14}$$

$$t^0 = 0, \quad x^0 = 0, \quad v^0 = v_0, \quad E^0 = E_0 \tag{15}$$

The Lagrange function (Lagrangian) of problem (12)–(15) for $\lambda_0 = 1$ can be written in the following form:

$$\Lambda = \xi T_0 + M_k (D - x^q)^2 + N_k \Delta\tau \cdot \xi \sum_{i=0}^{q} [(\max\{-E^i, 0\})^2 + (\max\{E^i - E_0, 0\})^2] +$$
$$+ \sum_{i=0}^{q-1} p_1^{i+1}(x^{i+1} - x^i - \Delta\tau\xi v^i) +$$
$$+ \sum_{i=0}^{q-1} p_2^{i+1}\left(v^{i+1} - v^i - \Delta\tau \cdot \xi\left(u^i - \begin{cases} 0{,}0037(v^i)^2, \; v^i \le 6 \text{ м/с}, \\ 0{,}6(v^i - 6) + 0{,}0037(v^i)^2, \; v^i > 6 \text{ м/с}, \end{cases}\right)\right) +$$
$$+ \sum_{i=0}^{q-1} p_3^{i+1}\left(E^{i+1} - E^i - \xi\Delta\tau\left(\gamma(E_1 - E^i) - \frac{u^i v^i}{\eta}\right)\right) + \sum_{i=0}^{q-1} p_4^{i+1}(t^{i+1} - t^i - \Delta\tau \cdot \xi),$$
$$i = \overline{0, q-1}$$

For a locally-optimal process $[\bar{t}, \bar{x}, \overline{E}, \bar{v}, \bar{u}, \bar{\xi}]$ there exist Lagrange multipliers $\lambda_0 \ge 0$ and p_j^i, $i = \overline{1, q}$, $j = 1, 2, 3, 4$ which are not simultaneously equal to zero and the following conditions are satisfied:

1) stationary condition with (within) $\lambda_0 = 1$:

$$\frac{\partial\overline{\Lambda}}{\partial x^i} = p_1^i - p_1^{i+1} = 0$$
$$\frac{\partial\overline{\Lambda}}{\partial v^i} = -p_1^{i+1}\xi\Delta\tau + p_2^i - p_2^{i+1} + \frac{u^i}{\eta} \cdot p_3^{i+1}\xi\Delta\tau + \left(\begin{cases} 0{,}0074 v^i, & \text{если } v^i \le 6 \text{ м/с} \\ 0{,}6 + 0{,}0074 v^i, & \text{если } v^i > 6 \text{ м/с} \end{cases}\right) \cdot p_2^{i+1}\xi\Delta\tau = 0 \tag{16}$$
$$\frac{\partial\overline{\Lambda}}{\partial E^i} = 2N_k\xi\Delta\tau\left(\begin{cases} -1, \text{ если } E^i < 0 \\ 0, \text{ если } E^i \ge 0 \end{cases} + \begin{cases} 1, \text{ если } E^i > E_0 \\ 0, \text{ если } E^i \le E_0 \end{cases}\right) + p_3^i - p_3^{i+1} + \gamma p_3^{i+1}\xi\Delta\tau = 0$$
$$\frac{\partial\overline{\Lambda}}{\partial t^i} = p_4^i - p_4^{i+1} = 0$$
$$i = \overline{0, q-1}$$

2) transversality condition for the right end:

$$\frac{\partial\overline{\Lambda}}{\partial x^q} = -2M_k(D - x^q) + p_1^q = 0, \quad \frac{\partial\overline{\Lambda}}{\partial v^q} = p_2^q = 0, \quad \frac{\partial\overline{\Lambda}}{\partial E^q} = p_3^q = 0, \quad \frac{\partial\overline{\Lambda}}{\partial t^q} = 1 + p_3^q = 0.$$

3) condition of a minimum of Langrangian (Lagrange function) on a control:

$$\overline{u^i} = \arg\min_{0 \le u^i \le U}\left\{(p_3^{i+1}\frac{v^i}{\eta} - p_2^{i+1})\Delta\tau\xi u^i\right\}, \quad i = \overline{0, q-1},$$

$$\bar{\xi} = \arg\min_{\xi\in[0;1]}\{\xi\Delta\tau(\frac{T_0}{\Delta\tau} + N_k\sum_{i=0}^{q}\left[\left(\max\left\{-E^i,0\right\}\right)^2 + \left(\max\left\{E^i - E_0, 0\right\}\right)^2\right]) -$$
$$-\sum_{i=0}^{q-1}\left[v^i p_1^{i+1} + p_2^{i+1}\left(u^i - r(v^i)\right) + p_3^{i+1}\left(d(E^i) - \frac{u^i v^i}{\eta}\right)\right]\}, \quad i = \overline{0, q-1} \tag{17}$$

We assume all the functions in the statement of the problem are convex in *u* for all *x.* From (15) we get recurrence formulas for the evaluation of conjugate functions:\

$$\begin{aligned}
&p_1^i = p_1^{i+1}, \quad p_1^q = 2M_k(D - x^q),\\
&p_2^i = p_1^{i+1}\xi\Delta\tau + p_2^{i+1} - \frac{u^i}{\eta}\cdot p_3^{i+1}\xi\Delta\tau -\\
&-\left(\begin{cases} 0,0074v^i, & \text{если } v^i \le 6 \text{ м/с}\\ 0,6 + 0,0074v^i, & \text{если } v^i > 6 \text{ м/с}\end{cases}\right)\cdot p_2^{i+1}\xi\Delta\tau, \quad p_2^q = 0,\\
&p_3^i = p_3^{i+1} - \gamma p_3^{i+1}\xi\Delta\tau -\\
&-2N_k\xi\Delta\tau\left(\begin{cases}-1, & \text{если } E^i < 0\\ 0, & \text{если } E^i \ge 0\end{cases} + \begin{cases}1, & \text{если } E^i > E_0\\ 0, & \text{если } E^i \le E_0\end{cases}\right), \quad p_3^q = 0,\\
&p_4^i = p_4^{i+1}, \quad p_4^q = -1,\\
&i = \overline{q-1, 0}.
\end{aligned} \tag{18}$$

3. THE ALGORITHM OF THE METHOD OF PENALTY FUNCTIONS

1) select the initial value of the penalty coefficient M_k, N_k, k=1;
2) set an initial approximation of controls:

$$[u]^0 = [u^{0(0)},...u^{q-1(0)}], \quad \text{where} \quad u^{i(0)} \in [0;U], i = \overline{0;q-1}, \xi \in [0;1];$$

3) construct the initial trajectory:

$$[x]^0 = [x^{0(0)},...x^{q(0)}], \quad [v]^0 = [v^{0(0)},...v^{q(0)}], \quad [E]^0 = [E^{0(0)},...E^{q(0)}], \quad [t]^0 = [t^{0(0)},...t^{q(0)}],$$

using difference equations (recurrence relations) and initial conditions:

$$\begin{cases}
x^{i+1(0)} = x^{i(0)} + \Delta\tau\cdot\xi v^{i(0)}, \quad x^{0(0)} = 0;\\
v^{i+1(0)} = v^{i(0)} + \Delta\tau\cdot\xi\left(u^{i(0)} - \begin{cases}0.0037\left(v^{i(0)}\right)^2, & v^{i(0)} \le 6 \text{ м/с},\\ 0,6(v^{i(0)} - 6) + 0.0037\left(v^{i(0)}\right)^2, & v^{i(0)} > 6 \text{ м/с}\end{cases}\right), \quad v^{0(0)} = v_0;\\
E^{i+1(0)} = E^{i(0)} + \Delta\tau\cdot\xi(\gamma E_1 - E^{i(0)}) - \frac{u^{i(0)}v^{i(0)}}{\eta}, \quad E^{0(0)} = E_0;\\
t^{i+1(0)} = t^{i(0)} + \Delta\tau\cdot\xi, \quad t^{0(0)} = 0;
\end{cases}$$
$$i = \overline{0;q-1};$$

4) calculate initial approximation of the objective function *I (0)* by the formula (12);

5) suppose: $I^{k^*} = I^{(0)}, [x]^{k^*} = [x]^{(0)}, [v]^{k^*} = [v]^{(0)}, [E]^{k^*} = [E]^{(0)}, [u]^{k^*} = [u]^{(0)}, [\xi]^{k^*} = [\xi]^{(0)}$ – k-th solution of the problem in the method of penalty functions;

6) move on to the next step of the method of penalty functions.

Increase the penalty factors ($\lambda > 1, \quad \mu > 1$ – constant multipliers):

$$M_{k+1} = \mu M_k, \quad N_{k+1} = \lambda N_k, \quad k := k+1;$$

7) calculate the conjugate functions by formulas (18);

8) calculate the control $[u]^{(1)} = [u^{0(1)},...u^{q-1(1)}], \quad \xi^{(1)}$:

$$u^{i(1)} = u^{i(0)} - \alpha \left[\frac{\partial \Lambda}{\partial u^i} \right]^{(0)} = u^{i(0)} - \alpha (p_3^{i+1,(0)} \frac{v^{i(0)}}{\eta} - p_2^{i+1,(0)}) \cdot \Delta \tau \cdot \xi,$$

$$\xi^{(1)} = \xi^{(0)} - \beta \left[\frac{\partial \Lambda}{\partial \xi} \right]^{(0)} = \xi^{(0)} - \beta \Delta \tau (\frac{T_0}{\Delta \tau} + N_k \sum_{i=0}^{q} \left[\left(\max\left\{ -E^i, 0 \right\} \right)^2 + \left(\max\left\{ E^i - E_0, 0 \right\} \right)^2 \right] -$$

$$\sum_{i=0}^{q-1} \left[v^i p_1^{i+1} + p_2^{i+1} \left(u^i - r\left(v^i\right) \right) + p_3^{i+1} \left(d(E^i) - \frac{u^i v^i}{\eta} \right) \right]),$$

and, if $u^{i(1)} < 0$, then $u^{i(1)} = 0$, if $u^{i(1)} > U$, then $u^{i(1)} = U$;
if $\xi(1) < 0$, then $\xi^{(1)} = 0$; if $\xi^{(1)} > 1$, then $\xi^{(1)} = 1$;

9) evaluate trajectory corresponding to this control

$$[x]^1 = [x^{0(1)},...x^{q(1)}], \quad [v]^1 = [v^{0(1)},...v^{q(1)}], \quad [E]^1 = [E^{0(1)},...E^{q(1)}], \quad [t]^1 = [t^{0(1)},...t^{q(1)}],$$

$$\begin{cases} x^{i+1(1)} = x^{i(1)} + \Delta\tau \cdot \xi v^{i(1)}, \quad x^{0(1)} = 0; \\ v^{i+1(1)} = v^{i(1)} + \Delta\tau \cdot \xi \left(u^{i(1)} - \begin{cases} 0.0037\left(v^{i(1)}\right)^2, & v^{i(1)} \le 6 \text{ м/с}, \\ 0,6(v^{i(1)} - 6) + 0.0037\left(v^{i(1)}\right)^2, & v^{i(0)} > 6 \text{ м/с} \end{cases} \right), \quad v^{0(1)} = v_0; \\ E^{i+1(1)} = E^{i(1)} + \Delta\tau \cdot \xi(\gamma E_1 - E^{i(1)}) - \dfrac{u^{i(1)} v^{i(1)}}{\eta}, \quad E^{0(1)} = E_0; \\ t^{i+1(1)} = t^{i(1)} + \Delta\tau \cdot \xi, \quad t^{0(1)} = 0; \end{cases}$$

10) calculate the next approximation of the objective function $I^{(1)}$ by the formula (12);

11) check the condition of monotonicity: if $I^{(1)} < I^{(0)}$, then go to 12), otherwise go to 8) and

$$\alpha := \frac{\alpha}{2}; \quad \beta := \frac{\beta}{2};$$

12) check whether the given accuracy of computing is achieved. If $\left| I^{(1)} - I^{(0)} \right| < \varepsilon_I^k$, then go to 14), otherwise go to 13);

13) suppose $I^{(0)} = I^{(1)}, \quad [x]^{(0)} = [x]^{(1)}, \quad [v]^{(0)} = [v]^{(1)}, \quad [E]^{(0)} = [E]^{(1)}, \quad [u]^{(0)} = [u]^{(1)}, \quad [\xi]^{(0)} = [\xi]^{(1)}$, go to 6);

14) suppose $I^{(k+1)^*} = I^{(1)}, [x]^{(k+1)^*} = [x]^{(1)}, [v]^{(k+1)^*} = [v]^{(1)}, [E]^{(k+1)^*} = [E]^{(1)}, [u]^{(k+1)^*} = [u]^{(1)}, [\xi]^{(k+1)^*} = [\xi]^{(1)}$ – (k+1) solution of the problem in the method of penalty functions.

4. RESULTS

We enter the following initial data: the distance of 1500 m, the energy: 2000 m^2/s^2; experimental time of 212 s; initial velocity of the athlete: 3 m/s. The distance is successfully passed by an athlete within 191,879 s – it is 20 s faster than the experimental time, coefficient of contraction of time =0,905; the first half of the race the athlete gains speed, then there is an active running, where the rate is close to the uniform, at the end of the race there is a finishing spurt; control is distributed similarly: first half of the race it is maximum, then it tends to zero (athlete rests, gaining strength for the final spurt), then control is maximum again; energy is actively expended in the first half of the distance, then there is a slight accumulation of it, and then again the expenditure follows. During the time of passing the distance an athlete spent about 1,100 m^2/s^2 of energy.

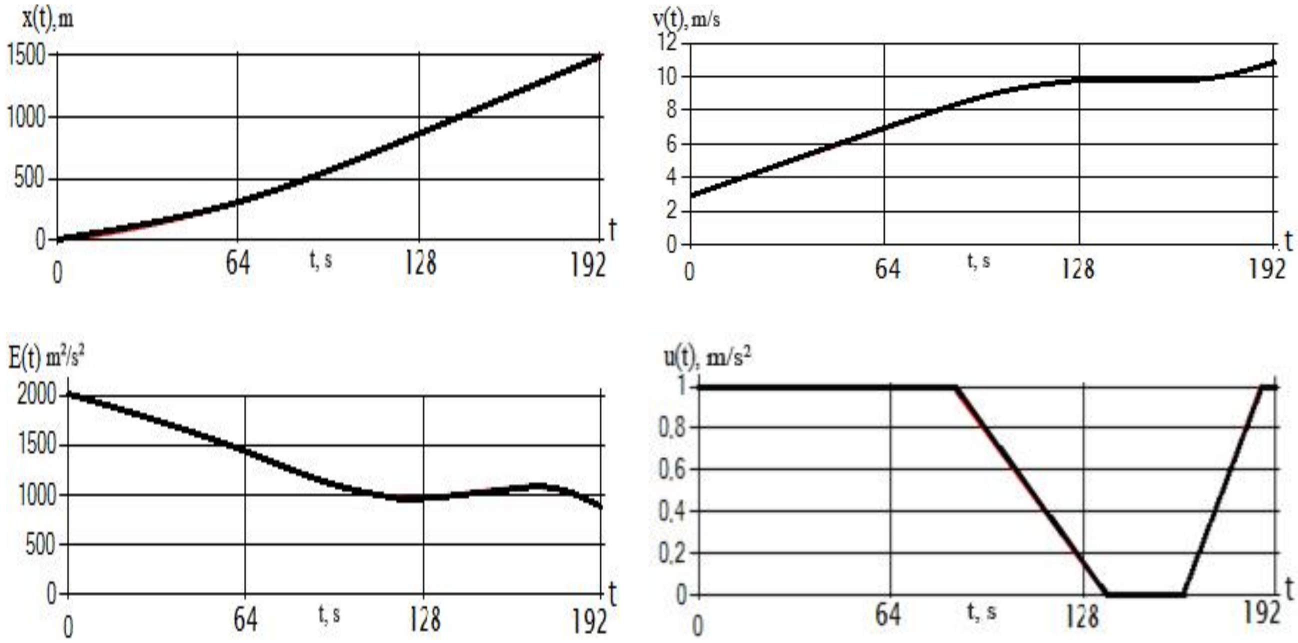

Fig. 1. Graphs of coordinate x(t), velocity v(t), energy expenditure E(t) and acceleration u(t) of an athlete on a distance

REFERENCES

1. Andreeva E.A., Tciruleva V.M. The calculus of variations and optimization techniques: the manual. Moscow: Higher School. 2006. 584 p.

2. Andreeva E.A., Semykina N.A. Optimal control: the manual. Tver: Tver branch of MESI. 2006. 264 p.

3. Behncke H. A mathematical model for the force and energetics in competitive running. J. Math. Biol. 1993. Vol. 31. Pp. 953–978.

ABOUT THE AUTHORS

Ovsyannikova Natalia Igorevna, graduated Pomorsky State University (Archangelsk) named after M.V. Lomonosov (1990), Ph.D. in physical and mathematical sciences, Associate Professor of the Department of Applied Mathematics, MSTU CA, author of 30 published scientific papers, sphere of scientific interests – mathematical modelling of complex dynamic systems, optimal control, numerical methods, calculus of variations, ordinary and stochastic differential equations, mathematical statistics, natmat68@mail.ru.

Popova Anna Andreevna, graduated Lomonosov Moscow State University, faculty of Mechanics and Mathematics (2014), Ph. D. student at the faculty of Economics (Lomonosov Moscow State University), sphere of scientific interests – calculus of variations, optimal control problems, mathematical and computer modelling, optimization methods.

ЗАДАЧА ОПТИМАЛЬНОГО УПРАВЛЕНИЯ В МОДЕЛИ БЕГА

Овсянникова Н.И., Попова А.А.

Статья посвящена поиску оптимального управления в модели бега на заданную дистанцию с целью минимизации времени. Управлением является сила, прикладываемая спортсменом (на единицу массы). Начальные условия и ограничения на управление заданы. В статье задача оптимального управления сводится к нахождению минимального времени забега путем введения коэффициента сжатия времени, который является вторым управляющим фактором.

Ключевые слова: модель соревнования в беге, оптимальное управление в задаче быстродействия, коэффициент сжатия времени.

СВЕДЕНИЯ ОБ АВТОРАХ

Овсянникова Наталья Игоревна, окончила Поморский государственный университет (Архангельск) имени М.В. Ломоносова (1990), кандидат физико-математических наук, доцент кафедры прикладной математики МГТУ ГА, автор 30 опубликованных научных работ, сфера научных интересов – математическое моделирование сложных динамических систем, оптимальное управление, численные методы, вариационное исчисление, обыкновенные и стохастические дифференциальные уравнения, математическая статистика, электронный адрес: natmat68@mail.ru.

Попова Анна Андреевна, окончила Московский государственный университет имени М.В. Ломоносова, механико-математический факультет (2014), аспирант факультета экономики МГУ имени М.В. Ломоносова, сфера научных интересов – вариационное исчисление, оптимальное управление, математическое и компьютерное моделирование, методы оптимизации.