ON SYMMETRY REDUCTIONS AND INVARIANT SOLUTIONS OF THE $k-\varepsilon$ TURBULENCE MODEL

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Methods of theoretical group analysis of differential equations are applied to the $k-\varepsilon$ turbulence model. Symmetry reductions of the $k-\varepsilon$ turbulence model with respect to some three-dimensional symmetry subalgebras are considered. Families of exact solutions are obtained.

Keywords: nonlinear differential equations, local infinitesimal symmetries, invariant solutions, the $k-\varepsilon$ turbulence model.

INTRODUCTION

The comprehension of turbulence is of fundamental interest from various aspects of human activities [1–6]. Although the turbulence problem has still remain an unsolved problem of classical physics, there exist various approaches to turbulence modeling. The $k-\varepsilon$ and other two-equation models are most widely used for engineering applications in spite of these models have several serious limitations [7, 8].

All two-equation models include the Navier-Stokes equations, which in principle can be integrated numerically [9]. However in many cases the computational efforts became enormous (see, for example, [5]), so numerical methods are viable for restricted class of problems.

Theoretical group analysis (symmetry methods) is a well known methodology to derive exact solutions of nonlinear differential equations. However, it is desirable to employ symmetry approach more systematically for development, improvement or calibration of turbulence models [10–13].

In this paper methods of theoretical group analysis of differential equations are applied to the $k-\varepsilon$ turbulence model. The classical symmetries of the $k-\varepsilon$ turbulence model have been calculated in [14]. The paper [14] contains also the complete ready-to-use list of symmetry subalgebras, which is necessary to construct invariant solutions of the equations under considerations. The aim of this paper is to obtain symmetry reductions and exact solutions of the $k-\varepsilon$ turbulence model. Here we shall not discuss physical meaning of obtained solutions [15].

The paper is organized as follows. In Section 2 we present the classical $k-\varepsilon$ turbulence model [7, 8, 14] and give a summary of the geometrical theory of nonlinear differential equations [16, 17]. In Section 3 we study invariance conditions for main classical symmetries of $k-\varepsilon$ turbulence model. In Section 4 we consider some reductions of $k-\varepsilon$ turbulence model with respect to three-dimensional subalgebras. In Section 5 we discussed methods of constructing solutions for reduction equations and obtain families of exact solutions for $k-\varepsilon$ turbulence model.

1. PRELIMINARIES

1.1. The $\boldsymbol{k-\varepsilon}$ turbulence model

The $k-\varepsilon$ turbulence model describes the motion of high Reynolds number turbulence flows and is derived from averages of Navier-Stokes equations by introducing the k- and ε-equations in order to obtain a closed set of equations [7, 8]:

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0, \tag{1}$$

$$\frac{\partial(\rho\,\bar{u}_j)}{\partial t} + \frac{\partial(\rho\,\bar{u}_i\bar{u}_j)}{\partial x_i} = -\frac{\partial \bar{p}}{\partial x_j} + \frac{\partial}{\partial x_i}\left(\bar{\tau}_{ij} - \rho\,\overline{u_i'u_j'}\right),\ j = 1,2,3, \tag{2}$$

$$\frac{\partial k}{\partial t}+\bar{u}_i\frac{\partial k}{\partial x_i}=\frac{\partial}{\partial x_i}\left\{\left(\nu+\frac{c_\mu}{\sigma_k}\frac{k^2}{\varepsilon}\right)\frac{\partial k}{\partial x_i}\right\}-\overline{u_i'u_j'}\frac{\partial \bar{u}_j}{\partial x_i}-\varepsilon, \tag{3}$$

$$\frac{\partial \varepsilon}{\partial t}+\bar{u}_i\frac{\partial \varepsilon}{\partial x_i}=\frac{\partial}{\partial x_i}\left\{\left(\nu+\frac{c_\mu}{\sigma_\varepsilon}\frac{k^2}{\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_i}\right\}-c_{\varepsilon_1}\frac{\varepsilon}{k}\overline{u_i'u_j'}\frac{\partial \bar{u}_j}{\partial x_i}-c_{\varepsilon_2}\frac{\varepsilon^2}{k}, \tag{4}$$

where $\bar{u}_i$ is the mean velocity component in the x_i direction, $\bar{p}$ is the mean pressure, k is the turbulence kinetic energy, ε is the rate dissipation of turbulence kinetic energy, $\nu = const$ is the viscosity, $\rho = const$ is the density,

$$-\overline{u_i'u_j'}=\frac{c_\mu k^2}{\varepsilon}\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)-\frac{2}{3}\delta_{ij}k,$$

is the Reynolds stress tensor, δ_{ij} beeing the Kronecker delta, $\bar{\tau}_{ij}=\rho\nu\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)$.

The five empirical constants c_μ, c_{ε_1}, c_{ε_2}, σ_k, σ_ε that appear in the equations are assigned the values: $c_\mu = 0.09$, $c_{\varepsilon_1} = 1.44 - 1.59$, $c_{\varepsilon_2} = 1.9 - 2.0$, $\sigma_k = 1.0$, $\sigma_\varepsilon = 1.3 - 1.47$. Throughout, for repeated indices the summation convention is used, the indices running from 1 to 3.

1.2. Symmetries and invariant solutions of partial differential equations

We expose here in a simplified, local coordinate form the basics of the geometrical approach to differential equations and its symmetries [16, 17].

Let $\mathcal{E} = \{F = 0\}$ be a system of differential equations given by

$$F_s\left(x,u,\dots,\frac{\partial^{|\sigma|}u^j}{\partial x_\sigma},\dots\right)=0,\ s=1,\dots,r,$$

where $u = (u^1,\dots,u^m)$ is the unknown vector-function in the variables $x = (x_1,\dots,x_n)$, $F = (F_1,\dots,F_r)$. In the framework of the geometrical theory any differential equation $\mathcal{E}$ of order k is considered as a submanifold in the space of k-jets $J^k(\pi)$ for some vector bundle $\pi: E^{n+m} \to M^n$. For example, system (1) – (4) is a submanifold $\mathcal{E} \subset J^2(\pi)$, where $\pi: D \times \mathrm{IR}^6 \to D$, $D \subseteq \mathrm{IR}^4(x_1,x_2,x_3,t)$, is the trivial bundle, $u^i = \bar{u}^i$, $i = 1,2,3$, $u^4 = \bar{p}$, $u^5 = k$, $u^6 = \varepsilon$ are the coordinates in IR^6.

Any infinitesimal symmetry of a system of differential equations has the form of evolutionary derivation

$$\bar{\ni}_\varphi=\sum_{\sigma,j}\bar{D}_\sigma\,(\varphi^j)\frac{\partial}{\partial u_\sigma^j},$$

where $\varphi = (\varphi^1,\dots,\varphi^m)$, $\varphi^i \in C^\infty(J^\infty(\pi))$, is the generating function of the symmetry, $D_i = \frac{\partial}{\partial x_i} + \sum_{j,\sigma} u_{\sigma i}^j \frac{\partial}{\partial u_\sigma^j}$ are the total derivative operators, $D_\sigma = D_{i_1} \circ \dots \circ D_{i_r}$ for $\sigma = (i_1,\dots,i_r)$, summation is taken over the internal coordinates on $\mathcal{E}$, the bar means the restriction to $\mathcal{E}$. We identify fields $\bar{\ni}_\varphi$ with their generating functions.

To find infinitesimal symmetries of the system $\mathcal{E}: F = 0$, $F = (F_1,\dots,F_r)$, $F_i \in C^\infty(J^k(\pi))$ one must solve the equation

$$\bar{l}_F(\varphi) = 0, \tag{5}$$

where l_F is the universal linearization operator

$$l_F = \left\| \sum_\sigma \frac{\partial F_i}{\partial u^j_\sigma} D_\sigma \right\|.$$

Let Sym $\mathcal{E}$ denote the vector space of all infinitesimal symmetries of an equation $\mathcal{E}$. We identify Sym $\mathcal{E}$ with the solution space of equation (5). The vector space Sym $\mathcal{E}$ is a Lie algebra with respect to the Jacobi bracket $\{\varphi, \psi\} = \bar{\ni}_\varphi(\psi) - \bar{\ni}_\psi(\varphi)$:

$$\{\varphi, \psi\}^j = \sum_{\sigma,\alpha} \left(\bar{D}_\sigma(\varphi^\alpha) \frac{\partial \psi^j}{\partial u^\alpha_\sigma} - \bar{D}_\sigma(\psi^\alpha) \frac{\partial \varphi^j}{\partial u^\alpha_\sigma} \right).$$

Let g be a subalgebra of the Lie algebra Sym $\mathcal{E}$. Then g-invariant solutions of $\mathcal{E}: F = 0$ are solutions of the system

$$F = 0,\ \varphi_1 = 0, \dots, \varphi_s = 0, \tag{6}$$

where φ_1, ... φ_s is a basis of g. System (6) is overdetermined and the fact that φ_i are symmetries means that the system is compatible. Under some regularity conditions, the problem of solving system (6) is equivalent to that of solving system with $n - s$ independent variables. We shall say that system (6) is the reduction of equation $\mathcal{E}$ with respect to the symmetry subalgebra $g =< \varphi_1, \dots, \varphi_s >$ (or with respect to the symmetries φ_1, ... φ_s). Note that in equation (6) one can use also generating functions of nonlocal symmetries [16, 18]. It is reasonable to use Reduce package [19] to solve equations (5), (6).

2. INVARIANCE CONDITIONS

In this section we solve the part $\varphi_1 = 0, \dots, \varphi_s = 0$ of system (6) for main classical symmetries of $k - \varepsilon$ turbulence model.

2.1. Generalized space translations and Galilean boost

Consider the following symmetries:

$$X_1(f) = (f u^1_1 - \dot{f},\ f u^2_1,\ f u^3_1,\ f p_1 + \rho \ddot{f} x_1,\ f k_1,\ f \varepsilon_1),$$
$$X_2(g) = (g u^1_2,\ g u^2_2 - \dot{g},\ g u^3_2,\ g p_2 + \rho \ddot{g} x_2,\ g k_2,\ g \varepsilon_2),$$
$$X_3(h) = (h u^1_3,\ h u^2_3,\ h u^3_3 - \dot{h},\ h p_3 + \rho \ddot{h} x_3,\ h k_3,\ h \varepsilon_3),$$

where f, g, h are arbitrary functions of t,

$$\{X_1(f), X_2(g)\} = \{X_2(g), X_3(h)\} = \{X_1(f), X_3(h)\} = 0,$$

while

$$\{X_i(f), X_i(g)\} = P(\ddot{f} g - f \ddot{g}),$$

where $P(h) = (0,0,0,h,0,0)$ [14].

When the functions f, g, h are being constant or linear, symmetries $X_1(f)$, $X_2(g)$, $X_3(h)$ are assigned to space translations or Galilean boost respectively.

For example, the system $X_3(h) = 0$ is of the form

$$h\frac{\partial \bar{u}^1}{\partial x_3} = 0, \quad h\frac{\partial \bar{u}^2}{\partial x_3} = 0, \quad h\frac{\partial \bar{u}^3}{\partial x_3} - \dot{h} = 0,$$
$$h\frac{\partial \bar{p}}{\partial x_3} + \rho\ddot{h}x_3 = 0, \quad h\frac{\partial k}{\partial x_3} = 0, \quad h\frac{\partial \varepsilon}{\partial x_3} = 0,$$

and its solutions are

$$\bar{u}^1 = \tilde{u}^1(t, x_1, x_2), \ \bar{u}^2 = \tilde{u}^2(t, x_1, x_2), \ \bar{u}^3 = \frac{\dot{h}}{h}x_3 + \tilde{u}^3(t, x_1, x_2),$$
$$\bar{p} = -\frac{\rho}{2}\frac{\ddot{h}}{h}x_3^2 + \tilde{p}(t, x_1, x_2), \ k = \tilde{k}(t, x_1, x_2), \ \varepsilon = \tilde{\varepsilon}(t, x_1, x_2).$$

In the same way one can solve system $X_1(f) = X_2(g) = X_3(h) = 0$ [20]:

$$\bar{u}^1 = \frac{\dot{f}}{f}x_1 + \alpha(t), \ \bar{u}^2 = \frac{\dot{g}}{g}x_2 + \beta(t), \ \bar{u}^3 = \frac{\dot{h}}{h}x_3 + \gamma(t),$$
$$\bar{p} = -\frac{\rho}{2}\left(\frac{\ddot{f}}{f}x_1^2 + \frac{\ddot{g}}{g}x_2^2 + \frac{\ddot{h}}{h}x_3^2\right) + p(t), \tag{7}$$
$$k = k(t), \ \varepsilon = \varepsilon(t).$$

2.2. Rotations

The symmetry algebra of the $k - \varepsilon$ turbulence model possesses three-dimensional rotation subalgebra $< R_{12}, R_{23}, R_{13} >$ [14], where

$$R_{ij} = \begin{pmatrix} x_j u_i^1 - x_i u_j^1 + \delta_{1j}u^i - \delta_{1i}u^j \\ x_j u_i^2 - x_i u_j^2 + \delta_{2j}u^i - \delta_{2i}u^j \\ x_j u_i^3 - x_i u_j^3 + \delta_{3j}u^i - \delta_{3i}u^j \\ x_j p_i - x_i p_j \\ x_j k_i - x_i k_j \\ x_j \varepsilon_i - x_i \varepsilon_2 \end{pmatrix}.$$

Consider, for example, the system $R_{12} = 0$:

$$x_2\frac{\partial \bar{u}^1}{\partial x_1} - x_1\frac{\partial \bar{u}^1}{\partial x_2} - u^2 = 0, \ x_2\frac{\partial \bar{u}^2}{\partial x_1} - x_1\frac{\partial \bar{u}^2}{\partial x_2} + u^1 = 0, \tag{8}$$

$$x_2\frac{\partial \bar{u}^3}{\partial x_1} - x_1\frac{\partial \bar{u}^3}{\partial x_2} = 0, \ x_2\frac{\partial \bar{p}}{\partial x_1} - x_1\frac{\partial \bar{p}}{\partial x_2} = 0, \tag{9}$$

$$x_2\frac{\partial k}{\partial x_1} - x_1\frac{\partial k}{\partial x_2} = 0, \ x_2\frac{\partial \varepsilon}{\partial x_1} - x_1\frac{\partial \varepsilon}{\partial x_2} = 0. \tag{10}$$

Equations (9)–(10) are of the form $yu_x - xu_y = 0$, $u = u(x,y)$. The latter equation possesses the solutions $u = f(x^2 + y^2)$.

System (8) (for a while we shall use the notations $u = \bar{u}^1$, $v = \bar{u}^2$, $x = x^1$, $y = x^2$)

$$\begin{cases} yu_x - xu_y = v, \\ yv_x - xv_y = -u \end{cases}$$

Is reduced to the heat equation

$$y^2 u_{xx} - 2xyu_{xy} + x^2 u_{yy} - xu_x - yu_y + u = 0,$$

which has the solution $u(x,y) = yf(x^2 + y^2) + xg(x^2 + y^2)$, hence $v(x,y) = yg(x^2 + y^2) - xf(x^2 + y^2)$.

So, we get the solution of the system (8)–(10):

$$\bar{u}^1 = x_2 f(x_1^2 + x_2^2, x_3, t) + x_1\, g(x_1^2 + x_2^2, x_3, t),$$
$$\bar{u}^2 = x_2\, g(x_1^2 + x_2^2, x_3, t) - x_1 f(x_1^2 + x_2^2, x_3, t),$$
$$\bar{u}^3 = \tilde{u}^3(x_1^2 + x_2^2, x_3, t),$$
$$\bar{p} = \tilde{p}(x_1^2 + x_2^2, x_3, t),$$
$$k = \tilde{k}(x_1^2 + x_2^2, x_3, t), \quad \varepsilon = \tilde{\varepsilon}(x_1^2 + x_2^2, x_3, t).$$

In the same way one can solve system $R_{12} = R_{23} = R_{13} = 0$ and get

$$\bar{u}^i = x_i \Phi(x_1^2 + x_2^2 + x_3^2, t), \tag{11}$$
$$\bar{p} = \tilde{p}(x_1^2 + x_2^2 + x_3^2, t), \tag{12}$$
$$k = \tilde{k}(x_1^2 + x_2^2 + x_3^2, t), \quad \varepsilon = \tilde{\varepsilon}(x_1^2 + x_2^2 + x_3^2, t). \tag{13}$$

2.3. Scale symmetry

Consider the generating function of scale symmetry [14]:

$$S = \begin{pmatrix} x_1 u_1^1 + x_2 u_2^1 + x_3 u_3^1 + 2tu_t^1 + u^1 \\ x_1 u_1^2 + x_2 u_2^2 + x_3 u_3^2 + 2tu_t^2 + u^2 \\ x_1 u_1^3 + x_2 u_2^3 + x_3 u_3^3 + 2tu_t^3 + u^3 \\ x_1 p_1 + x_2 p_2 + x_3 p_3 + 2tp_t + 2p \\ x_1 k_1 + x_2 k_2 + x_3 k_3 + 2tk_t + 2k \\ x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 + 2t\varepsilon_t + 4\varepsilon \end{pmatrix}.$$

All equations of the system $S = 0$ are of the form

$$x_1 ux_1 + x_2 ux_2 + x_3 ux_3 + 2tut + au = 0, \tag{14}$$

where $u = \bar{u^\iota}$, $\bar{p}$, k or ε, $a = 1,2,4$.

Using the characteristic equation for (14)

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \frac{dx_3}{x_3} = \frac{dt}{2t} = \frac{du}{-au},$$

we get the solution of the system $S = 0$ in the form

$$\bar{u}^i = \frac{1}{\sqrt{t}}\,\tilde{u}^i\left(\frac{x_1^2}{t}, \frac{x_2^2}{t}, \frac{x_3^2}{t}\right),\ \bar{p} = \frac{1}{t}\,\tilde{p}\left(\frac{x_1^2}{t}, \frac{x_2^2}{t}, \frac{x_3^2}{t}\right),$$

$$k = \frac{1}{t}\,\tilde{k}\left(\frac{x_1^2}{t}, \frac{x_2^2}{t}, \frac{x_3^2}{t}\right), \quad \varepsilon = \frac{1}{t^2}\,\tilde{\varepsilon}\left(\frac{x_1^2}{t}, \frac{x_2^2}{t}, \frac{x_3^2}{t}\right).$$

3. REDUCTIONS OF THE $k - \varepsilon$ TURBULENCE MODEL

In this section we consider three reductions of the $k - \varepsilon$ turbulence model with respect to three-dimensional subalgebras. Let us remark that the complete list of three-dimensional symmetry subalgebras, which is presented in the paper [14], contains more than 20 items. Calculations show that reductions under considerations involves ordinary or partial differential equations only for functions k and ε, while for other four function expression containing arbitrary constants (and function of t in case of $\bar{p}$) are obtained. This means that regularity conditions mentioned at the end of Section 2.2 in case of the $k - \varepsilon$ turbulence model are broken.

1. The reduction with respect to subalgebra $< X_1(f), X_2(g), X_3(h) >$.

Combining (7) and (1)–(4), we obtain (see [20])

$$\dot{k} = L\,\frac{k^2}{\varepsilon} - \varepsilon, \tag{15}$$

$$\dot{\varepsilon} = c_{\varepsilon_1} L k - c_{\varepsilon_2}\,\frac{\varepsilon^2}{k}, \tag{16}$$

where

$$L = 2c_\mu\left(\left(\frac{\dot{f}}{f}\right)^2 + \left(\frac{\dot{g}}{g}\right)^2 + \left(\frac{\dot{h}}{h}\right)^2\right), \tag{17}$$

while

$$\bar{u}^1 = \frac{\dot{f}x_1 + a_1}{f}, \bar{u}^2 = \frac{\dot{g}x_2 + a_2}{g}, \bar{u}^3 = \frac{\dot{h}x_3 + a_3}{h}, \ \ fgh = a, \tag{18}$$

$$\bar{p} = -\frac{\rho}{2}\left(\frac{\ddot{f}}{f}x_1^2 + \frac{\ddot{g}}{g}x_2^2 + \frac{\ddot{h}}{h}x_3^2\right) + p(t),\ k = k(t),\ \varepsilon = \varepsilon(t), \tag{19}$$

where a, a_i are constants.

2. The reduction with respect to subalgebra $< X_1(f), X_2(f), R_{12} >$.

Combining equations $X_1(f) = X_2(f) = R_{12} = 0$ and (1), we get

$$\bar{u}^1 = \frac{\dot{f}}{f}x_1,\ \bar{u}^2 = \frac{\dot{f}}{f}x_2,\ \bar{u}^3 = -2\frac{\dot{f}}{f}x_3 + \alpha(t), \tag{20}$$

$$\bar{p} = -\frac{\rho\ddot{f}}{2f}(x_1^2 + x_2^2) + p(x_3, t), \tag{21}$$

$$k = k(x_3, t),\ \varepsilon = \varepsilon(x_3, t). \tag{22}$$

Substituting (20), (21) in (2), we obtain

$$\frac{1}{\rho}\bar{p} = \left(\frac{\ddot{f}}{f} - 3\left(\frac{\dot{f}}{f}\right)^{\cdot}\right)x_3^2 + \left(\dot{\alpha} - 2\alpha\frac{\dot{f}}{f}\right)x_3 - 2c_\mu\frac{\dot{f}}{f}\frac{k^2}{\varepsilon} - \frac{2}{3}k + p(t).$$

Finally, using (3), (4) and (22), we get the reduction

$$k_t + u^3 k_3 = D_3\left(\left(\nu + \frac{c_\mu}{\sigma_k}\frac{k^2}{\varepsilon}\right)k_3\right) + 12\left(\frac{\dot{f}}{f}\right)^2\frac{k^2}{\varepsilon} - \varepsilon,$$

$$\varepsilon_t + u^3\varepsilon_3 = D_3\left(\left(\nu + \frac{c_\mu}{\sigma_\varepsilon}\frac{k^2}{\varepsilon}\right)\varepsilon_3\right) + 12c_{\varepsilon_1}\left(\frac{\dot{f}}{f}\right)^2 k - c_{\varepsilon_2}\frac{\varepsilon^2}{k}.$$

3. The reduction with respect to subalgebra $< R_{12}, R_{23}, R_{13} >$.
Substituting (11) in (1), we obtain

$$\bar{u}^i = A(t)x_i R^{-\frac{3}{2}},\ \ R = x_1^2 + x_2^2 + x_3^2.$$

Using (2) and (12), one get

$$\frac{1}{\rho}\bar{p} = \frac{1}{3}A^2R^{-2} - \frac{1}{3}A_tR^{-1/2} - \frac{2}{3}k - 4c_\mu\int R^{-3/2}\left(\frac{k^2}{\varepsilon}\right)_R^{\prime} dR.$$

Finally, combining (3), (4) and (13), we obtain the reduction system for functions $k = k(t, R)$ and $\varepsilon = \varepsilon(t, R)$:

$$k_t + 2AR^{-1/2}k_R = 2R\left(\left(\nu + \frac{c_\mu}{\sigma_k}\frac{k^2}{\varepsilon}\right)k_R\right)_R^{\prime} + 6\left(\nu + \frac{c_\mu}{\sigma_k}\frac{k^2}{\varepsilon}\right)k_R + 12c_\mu A^2R^{-3}\frac{k^2}{\varepsilon} - \varepsilon,$$

$$\varepsilon_t + 2AR^{-1/2}\varepsilon_R = 2R\left(\left(\nu + \frac{c_\mu}{\sigma_\varepsilon}\frac{k^2}{\varepsilon}\right)\varepsilon_R\right)_R^{\prime} + 6\left(\nu + \frac{c_\mu}{\sigma_\varepsilon}\frac{k^2}{\varepsilon}\right)\varepsilon_R + 12c_\mu c_{\varepsilon_1}A^2R^{-3}k - c_{\varepsilon_2}\frac{\varepsilon^2}{k}.$$

4. INVARIANT SOLUTIONS

Reductions 2 and 3 in Section 4 are complicated systems of nonlinear partial differential equations for functions k and ε. To find solutions of these system one can use symmetry method again or try to solve the systems numerically (cf. [12], [13]). This will be discussed elsewhere.

For Reductions 1 in paper [20] families of exact solutions were obtained.

For example, if $X_1(f)$, $X_2(g)$, $X_3(h)$ are translations ($f = g = h = 1$, $L = 0$) we get invariant solution:

$$\bar{u}^1 = a_1,\ \bar{u}^2 = a_2,\ \bar{u}^3 = a_3,\ \bar{p} = p(t),$$
$$k = (C_1 t + C_2)^{\frac{1}{1-c_{\varepsilon_2}}},\quad \varepsilon = \frac{C_1}{c_{\varepsilon_2} - 1}(C_1 t + C_2)^{\frac{c_{\varepsilon_2}}{1-c_{\varepsilon_2}}}.$$

where a_i, C_1, C_2 are arbitrary constants.

Also in [20] it was shown, that in the case $L \neq 0$ system (15)–(16) transforms to

$$\dot{\varepsilon} = \left(c_{\varepsilon_1} L w - c_{\varepsilon_2} w^{-1}\right)\varepsilon,\ \ k = w\varepsilon, \tag{23}$$

where

$$\dot{w} = -l(t)w^2 + a, \tag{24}$$
$$l(t) = (c_{\varepsilon_1} - 1)L \geq 0,\ \ a = c_{\varepsilon_2} - 1 > 0. \tag{25}$$

The substitution $w = \frac{1}{l}\frac{\dot{v}}{v}$ transforms equation (24) to linear equation

$$l\ddot{v} - \dot{l}\dot{v} = al^2 v. \tag{26}$$

Integrating (23), one get

$$\ln\,|\varepsilon| = \int (c_{\varepsilon_1} L w - c_{\varepsilon_2} w^{-1})dt = \frac{c_{\varepsilon_1}}{c_{\varepsilon_1} - 1}\int l \cdot \frac{1}{l}\frac{\dot{v}}{v}dt - c_{\varepsilon_2}\int \frac{lv}{\dot{v}}dt =$$
$$= \frac{c_{\varepsilon_1}}{c_{\varepsilon_1} - 1}\ln\,|v| - c_{\varepsilon_2}\int \frac{l\ddot{v} - \dot{l}\dot{v}}{al\dot{v}}dt =$$
$$= \frac{c_{\varepsilon_1}}{c_{\varepsilon_1} - 1}\ln\,|v| - \frac{c_{\varepsilon_2}}{c_{\varepsilon_2} - 1}(\ln\,|\dot{v}| - \ln\,|l|) + ln|C|.$$

Therefore, we have

$$\varepsilon = Cv^{\frac{c_{\varepsilon_1}}{c_{\varepsilon_1}-1}}\left(\frac{l}{\dot{v}}\right)^{\frac{c_{\varepsilon_2}}{c_{\varepsilon_2}-1}},\ \ k = w\varepsilon = \frac{C}{l}\frac{\dot{v}}{v}v^{\frac{c_{\varepsilon_1}}{c_{\varepsilon_1}-1}}\left(\frac{l}{\dot{v}}\right)^{\frac{c_{\varepsilon_2}}{c_{\varepsilon_2}-1}} = Cv^{\frac{1}{c_{\varepsilon_1}-1}}\left(\frac{l}{\dot{v}}\right)^{\frac{1}{c_{\varepsilon_2}-1}}. \tag{27}$$

These results can be summarize as follows. To find invariant solutions in the case $L \neq 0$ we must choose functions f, g, h such that $fgh = A = const$ and find $\bar{u}^i$, $\bar{p}$ by (18), (19), then calculate function $l(t)$ (see (17), (25)), solve equation (26) and find function v. Then, using (27), we get ε and k. Thus, the main problem in this procedure is to solve equation (26). Below we consider two cases when this equation can be integrated.

In the case $f = b_1 e^{\lambda t}$, $g = b_2 e^{\mu t}$, $h = b_3 e^{-(\lambda+\mu)t}$, where λ, μ, b_i are arbitrary constants, $l = const$, $v = C_3 e^{\sqrt{al}\,t} + C_4 e^{-\sqrt{al}\,t}$ we get:

$$\bar{u}^1 = \lambda x_1 + a_1 e^{-\lambda t},\ \bar{u}^2 = \mu x_2 + a_2 e^{-\mu t},\ \bar{u}^3 = -(\lambda+\mu)x_1 + a_3 e^{(\lambda+\mu)t},$$
$$\bar{p} = -\frac{\rho}{2}(\lambda^2 x_1^2 + \mu^2 x_2^2 + (\lambda+\mu)^2 x_3^2) + p(t),$$
$$\varepsilon = C_5 \frac{\left(C_3 e^{\sqrt{al}\,t} + C_4 e^{-\sqrt{al}\,t}\right)^{\frac{c_{\varepsilon_1}}{c_{\varepsilon_1}-1}}}{\left(C_3 e^{\sqrt{al}\,t} - C_4 e^{-\sqrt{al}\,t}\right)^{\frac{c_{\varepsilon_2}}{c_{\varepsilon_2}-1}}},\ k = C_5\sqrt{\frac{a}{l}} \frac{\left(C_3 e^{\sqrt{al}\,t} + C_4 e^{-\sqrt{al}\,t}\right)^{\frac{1}{c_{\varepsilon_1}-1}}}{\left(C_3 e^{\sqrt{al}\,t} - C_4 e^{-\sqrt{al}\,t}\right)^{\frac{1}{c_{\varepsilon_2}-1}}}$$

(a_i, C_i are arbitrary constants).

In the case $f = t^\alpha$, $\alpha \neq 0$, $g = 1/f$, $h = const$, equation (26) is the Euler equation

$$t^2\ddot{v} + 2t\dot{v} - 4c_\mu(c_{\varepsilon_1}-1)(c_{\varepsilon_2}-1)\alpha^2 v = 0$$

with the solution $v = C_6 t^{\mu-\frac{1}{2}} + C_7 t^{-\mu-\frac{1}{2}}$, where

$$\mu = \frac{1}{2}\sqrt{1 + 16c_\mu(c_{\varepsilon_1}-1)(c_{\varepsilon_2}-1)\alpha^2}.$$

So one get family of invariant solutions

$$\bar{u}^1 = \alpha x_1/t + a_1 t^{-\alpha}, \bar{u}^2 = -\alpha x_2/t + a_2 t^\alpha, \bar{u}^3 = a_3,$$
$$\bar{p} = -\frac{\alpha\rho}{2}\cdot\frac{(\alpha-1)x_1^2 + (\alpha+1)x_2^2}{t^2} + p(t),$$
$$k = \frac{C_8 c_\alpha^{\frac{1}{c_{\varepsilon_2}-1}}\left(C_6 t^{\mu-\frac{1}{2}} + C_7 t^{-\mu-\frac{1}{2}}\right)^{\frac{1}{c_{\varepsilon_1}-1}}}{\left(C_6(\mu-1/2)t^{\mu+\frac{1}{2}} - C_7(\mu+1/2)t^{-\mu+\frac{1}{2}}\right)^{\frac{1}{c_{\varepsilon_2}-1}}},$$
$$\varepsilon = \frac{C_8 c_\alpha^{\frac{c_{\varepsilon_2}}{c_{\varepsilon_2}-1}}\left(C_6 t^{\mu-\frac{1}{2}} + C_7 t^{-\mu-\frac{1}{2}}\right)^{\frac{c_{\varepsilon_1}}{c_{\varepsilon_1}-1}}}{\left(C_6(\mu-1/2)t^{\mu+\frac{1}{2}} - C_7(\mu+1/2)t^{-\mu+\frac{1}{2}}\right)^{\frac{c_{\varepsilon_2}}{c_{\varepsilon_2}-1}}},$$

where $c_\alpha = 4c_\mu(c_{\varepsilon_1}-1)(c_{\varepsilon_2}-1)\alpha^2$, while a_i, C_i are arbitrary constants.

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О РЕДУКЦИЯХ И ИНВАРИАНТНЫХ РЕШЕНИЯХ МОДЕЛИ $k-\varepsilon$ ТУРБУЛЕНТНОСТИ

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Методы группового анализа дифференциальных уравнений применяются к модели $k-\varepsilon$ турбулентности. Рассмотрены редукции модели $k-\varepsilon$ турбулентности по отношению к трехмерной подалгебре симметрий. Получены семейства точных решений.

Ключевые слова: нелинейные дифференциальные уравнения, локальные симметрии, инвариантные решения, модель $k-\varepsilon$ турбулентности.

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