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SYMMETRIES AND LAX INTEGRABILITY OF THE GENERALIZED PROUDMAN-JOHNSON EQUATION

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We study local symmetries of the generalized Proudman-Johnson equation. Symmetries of a partial differential equation may be used to find its invariant solutions. In particular, if φ is a characteristic of a symmetry for a PDE $H = 0$ then the φ-invariant solution of the PDE is a solution to the compatible over-determined system $H = 0$, $\varphi = 0$. We show that the Lie algebra of local symmetries for the generalized Proudman-Johnson equation is infinite-dimensional. Reductions of equation with respect to the local symmetries provide ordinary differential equations that describe invariant solutions. For a certain value of the parameter entering the equation we find some cases when the reduced ODE is integrable by quadratures and thus allows one to construct exact solutions. Differential coverings (or Wahlquist-Estabrook prolongation structures, or zero-curvature representations, or integrable extensions, etc.) are of great importance in geometry of PDEs. The theory of coverings is a natural framework for dealing with inverse scattering constructions for soliton equations, Bäcklund transformations, recursion operators, nonlocal symmetries and nonlocal conservation laws, Darboux transformations, and deformations of nonlinear PDEs. In the last section we show that in the case of a certain value of the parameter entering the equation it has a differential covering. This property is referred to as a Lax integrability.

Key words: generalized Proudman-Johnson equation, local symmetry of differential equation, invariant solution, differential covering, Bäcklund transformation.

INTRODUCTION

In this paper we consider the generalized Proudman-Johnson equation, [1, 2, 3, 4], in its potential form

$$u_{tx} = u\, u_{xx} + \alpha\, u_x^2 + \varepsilon\, u_{xxx}. \tag{1}$$

Equation (1) with $\alpha = -1$ was derived from the Navier-Stokes equations for incompressible viscous fluid by assuming a special similarity form on the velocity field; see [1] and the references therein. When $\varepsilon = 0$, equation (1) gets the form of the generalized Hunter–Saxton equation

$$u_{tx} = u\, u_{xx} + \alpha\, u_x^2. \tag{2}$$

Thus equation (1) may be considered as a dispersive deformation of (2). In [5] it was shown that equation (2) is linearizable and integrable by quadratures, see also [6]. In the present paper we consider case $\varepsilon \neq 0$. In this case after the scaling $u \mapsto \varepsilon\, u$, $t \mapsto \varepsilon^{-1}\, t$ equation (1) acquires the form

$$u_{tx} = u\, u_{xx} + \alpha\, u_x^2 + u_{xxx}. \tag{3}$$

We show that the Lie algebra of local symmetries for equation (3) is infinite-dimensional. Reductions of equation (3) with respect to the local symmetries provide ordinary differential equations that describe invariant solutions of (3). For $\alpha = 1$ we find some cases when the reduced ODE is integrable by quadratures and thus allows one to construct exact solutions of (3).

Finally, we show that in the case of $\alpha = 2$ equation (3) has a differential covering, [7, 8, 9]. This property is referred to as a Lax integrability. The Lax integrability of (2) in the case of $\alpha = \frac{1}{2}$ was established in [10], see also [11].

We follow definitions and notation of [7, 8, 9].

1. LOCAL SYMMETRIES

Local symmetries of equation (3) are solutions $\varphi = \varphi(t, x, u, u_t, u_x)$ to its linearization

$$\bar{D}_t \bar{D}_x(\varphi) = u\, \bar{D}_x^2(\varphi) + u_{xx}\, \varphi + 2\, \alpha\, u_x\, \bar{D}_x(\varphi) + \bar{D}_x^3 \varphi),$$

where $\bar{D}_t$ and $\bar{D}_x$ are restictions of the total derivatives $D_t = \partial/\partial_t + \sum u_{i+1,j}\, \partial/\partial u_{i,j}$ and $D_x = \partial/\partial_x + \sum u_{i,j+1}\, \partial/\partial u_{i,j}$ on (3). Here and below we denote $u_{i,j} = u_{\underbrace{t,\dots,t}_{i\, times}\, ,\, \underbrace{x,\dots,x}_{j\, times}} = \frac{\partial^{i+j} u}{\partial t^i \partial x^j}$.

The direct computations give the following result.

THEOREM 1. *The local symmetry algebra of equation is infinite-dimensional. In the case of* $\alpha \neq 1$ *it is generated by the vector fields with characteristics*

$$\varphi(A) = A\, u_x + A',\ \psi_0 = u_t,\ \psi_1 = 2\, t\, u_t + x\, u_x + u,$$

where $A = A(t)$ *and* $B = B(t)$ *below are arbitrary functions of* t*, and primes denote derivatives of functions of one argument w.r.t. their arguments. The structure of a Lie algebra is given by the relations*

$$\{\varphi(A), \varphi(B)\} = 0,\ \{\psi_0, \varphi(A)\} = -\varphi(A'),\ \{\psi_1, \varphi(A)\} = -\varphi(A - 2\, t\, A'),\ \{\psi_0, \psi_1\} = -2\, \psi_0.$$

In the case of $\alpha = 1$ *the symmetry algebra is extended by the vector field with the characteristic*

$$\psi_2 = t^2\, u_t + t\, x\, u_x + t\, u + x.$$

The additional relations of the extended Lie algebra read

$$\{\psi_2, \varphi(A)\} = -\varphi(t\, A - t^2\, A'),\ \{\psi_0, \psi_2\} = -\psi_1,\ \{\psi_1, \psi_2\} = -2\, \psi_2.$$

2. INVARIANT SOLUTIONS

Symmetries of a partial differential equation may be used to find its invariant solutions. In particular, if φ is a characteristic of a symmetry for a PDE $H(t, x, u, u_t, u_x, \dots, u_{i,j}) = 0$, then the φ-invariant solution of the PDE is a solution to the compatible over-determined system $H = 0$, $\varphi = 0$.

For equation (3) we consider ψ_1-invariant solutions. The condition $\psi_1 = 2\, t\, u_t + x\, u_x + u = 0$ holds whenever

$$u = t^{-1/2}\, w(z), \qquad z = x\, t^{-1/2}, \tag{4}$$

where w is an arbitrary function of z.

Substituting for (4) into (3) yields the following ODE:

$$w''' + \left(w + \frac{1}{2}\, z\right) w'' + \alpha\, (w')^2 + w' = 0. \tag{5}$$

It is convenient to change the dependent variable in (5) via the transformation $w(z) = r(z) - \frac{1}{2}\, z$, then we have

$$r''' + r\,r'' + \alpha\,(r')^2 + (\alpha - 1)\,r' - \frac{\alpha - 2}{4} = 0. \tag{6}$$

We note that in the case of $\alpha = 1$ the order of this equation may be reduced by two. Indeed, in this case equation (6) has the form

$$\left(r' + \frac{r^2}{2} - \frac{z^2}{8}\right)'' = 0,$$

This equation may be integrated twice, then up to the change of the independent variable $z \mapsto z + c$ it obtaines the form of Riccati's equation

$$r' = -\frac{1}{2}\,r^2 + \frac{z^2}{8} + \beta, \tag{7}$$

where β is an arbitrary constant. Then substituting for $r(z) = 2\,q'(z)/q(z)$ into (7) yields Weber's equation, [13, 14],

$$q'' = \frac{1}{16}\,(z^2 + 8\,\beta)\,q. \tag{8}$$

The scaling $z = \sqrt{2}\,s$ of the independent variable gives the normal form

$$\frac{d^2 v}{ds^2} = \left(\frac{1}{4}\,s^2 + \beta\right)\,v \tag{9}$$

of equation (8) with $v(s) = q(z)$. It was proven in [15] that equation (9) is integrable by quadratures whenever $\beta = \frac{1}{2} + n$ with $n \in \mathbb{Z}$. Therefore for any β of this form we may integrate equation (7) and then find function $w(z)$ that defines the invariant solution (4) of equation (3) with $\alpha = 1$. For example, in the cases $n = 0$ and $n = 1$ we have, respectively,

$$w = \frac{2\ exp\,(-\frac{1}{4}\,z^2)}{\Phi(\frac{1}{2}\,z) + C}$$

and

$$w = \frac{4\,\left(\Phi(\frac{1}{2}\,z) + C\right)}{z\,(\Phi(\frac{1}{2}\,z) + C) + 4\ exp\,(-\frac{1}{4}\,z^2)}$$

where C is an arbitrary constant and

$$\Phi(\zeta) = \int_0^{\zeta} \exp\,(-\tau^2)\,d\tau.$$

LAX INTEGRABILITY IN THE CASE OF $\alpha = 2$

Differential coverings (or Wahlquist-Estabrook prolongation structures, [16], or zero-curvature representations, [17], or integrable extensions, [18], etc.) are of great importance in geometry of PDEs. The theory of coverings is a natural framework for dealing with inverse scattering constructions for soliton equations, Bäcklund transformations, recursion operators, nonlocal symmetries and nonlocal conservation laws, Darboux transformations, and deformations of nonlinear PDEs, [7, 8, 9]. In this section we show that in the case of $\alpha = 2$ equation (3) admits a covering. We have the following result, whose proof is a direct computation:

THEOREM 2. *System*

$$\begin{cases} q_x = q^2 + \frac{1}{2} u_x, \\ q_t = u\, q^2 + u_x\, q + \frac{1}{2} (u_{xx} + u\, u_x) \end{cases} \tag{10}$$

defines a differential covering for equation (3) with $\alpha = 2$.

The structure algebra of this covering is generated by the vector fields ∂_q, $q\,\partial_q$, $q^2\,\partial_q$, and is therefore isomorphic to the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$.

Excluding u from (10) yields equation

$$q_{tx} = q_{xxx} + 2\, q_x^2 - 8\, q^2\, q_x + q_{xx}\, q_x^{-1} (q_t + 2\, q^3 - q_{xx}). \tag{11}$$

In other words, system (10) defines a Bäcklund transformation between equations (3) and (11).

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СИММЕТРИИ И ИНТЕГРИРУЕМОСТЬ ПО ЛАКСУ ОБОБЩЕННОГО УРАВНЕНИЯ ПРУДМАНА – ДЖОНСОНА

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Изучаются локальные симметрии обобщённого уравнения Прудмана – Джонсона. Симметрии дифференциального уравнения в частных производных могут использоваться для нахождения его инвариантных решений. В частности, если φ есть производящая функция симметрии для уравнения $H = 0$, то $\varphi -$ инвариантные решения суть решения переопределённой совместной системы $H = 0, \varphi = 0$. Показано, что алгебра Ли локальных симметрий обобщённого уравнения Прудмана-Джонсона является бесконечной. Найдены некоторые случаи, когда редуцированное при помощи симметрий уравнение сводится к обыкновенным дифференциальным уравнениям, которые интегрируются в квадратурах, что позволяет построить соответствующие точные решения. Дифференциальные накрытия (или структуры продолжения Волквиста – Истабрука, или представления нулевой кривизны, или интегрируемые расширения и так далее) весьма важны в геометрии уравнений в частных производных. Теория дифференциальных накрытий есть естественный язык для работы с обратной задачей теории рассеивания в случае солитонных уравнений, преобразований Бэклунда, операторов рекурсии, нелокальных симметрий и нелокальных законов сохранения, преобразований Дарбу и деформаций нелинейных уравнений. В последнем разделе статьи показано, что при некоторых значениях параметра, входящего в уравнение Прудмана – Джонсона, оно обладает дифференциальным накрытием. Это свойство также называется интегрируемостью по Лаксу.

Ключевые слова: обобщённое уравнение Прудмана – Джонсона, дифференциальное накрытие, локальные симметрии, инвариантные решения, преобразование Бэклунда.

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