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SAWTOOTH SOLUTIONS TO THE BURGERS EQUATION ON AN INTERVAL

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The asymptotic behavior of solutions of the Burgers equation and its generalizations with initial value - boundary problem on a finite interval with constant boundary conditions is studied. Since the equation describes the movement in a dissipative medium, the initial profile of the solution will evolve to an time-invariant solution with the same boundary values. However there are three ways of obtaining the same result: the initial profile may regularly decay to the smooth invariant solution; or a Heaviside-type gap develops through a dispersive shock and multi-oscillations; or an asymptotic limit is a stationary 'sawtooth' solution with periodical breaks of derivative.

Key words: Burgers equation, initial value - boundary problem, gradient catastrophe, sawtooth solutions.

1. Introduction

The Burgers equation

$$u_t(x,t) = \varepsilon^2 u_{xx}(x,t) - \alpha \cdot u(x,t)\, u_x(x,t). \tag{1}$$

is related to the viscous medium whose oscillations it describes. This viscosity dampens oscillations except for stationary solutions which are invariant for some subalgebra of the full symmetry algebra of the equation. While studying the equation on the whole line only bounded solutions are usually taken into account since only they have a physical meaning. It is not the case for a finite interval as an unbounded solution may still remain bounded within an interval. Thus we obtain a wider choices of invariant solutions and asymptotics and, consequently, some new effects.

We consider initial value - boundary problem (IVBP) for the Burgers equation on a finite interval:

$$u(x,0) = f(x); \quad u(a,t) = l(t); \quad u(b,t) = r(t); \quad x \in [a,b]. \tag{2}$$

The case of constant boundary conditions $u(a,t) = A, \quad u(b,t) = B$ and related asymptotics are of a special interest here.

Some of our results are similar to those of Dubrovin*et al* [1; 2; 3] dealing with a formation of dispersive shocks in a class of Hamiltonian dispersive regularizations of the quasi-linear transport equation. For the Burgers equation the shocks resulting in breaks (and preceded by a multi-oscillation) do develops for some IVBPs; some other IVBPs lead to a monotonic convergence to an invariant solutions. One more possibility for the asymptotics is a class of periodic 'saw-tool' profile solutions. Such profiles (though for travelling waves on a line) are known in nonlinear acoustics [4; 5]; they form in media where nonlinearity dominates over dispersion, diffraction and absorption. The short history of this research is as follows.

The Burgers equation has been used by many authors (Lighthill - 1956, Soluyan&Khokhlov - 1961, Blackstock - 1964) to describe the propagation of one-dimensional acoustic signals of moderate amplitude. Here, *-u* denotes the velocity or the excess density; if the Lagrangian coordinate *Y* measures distance from the driving piston, and, if *t* denotes time and c_0 the linearized sound speed, then $x = c_0 t - Y$. Thus x/c_0 denotes the time elapsed since the passage of a reference wavelet while the constants α and ε^2 quantify the effects of amplitude dispersion and of diffusion respectively, and the subscripts denote partial differentiation.

It is, of course, widely known that the transformation due to Hopf (1940) and Cole (1951) reduces (1) to the heat conduction equation. Using this transformation, Lighthill (1956) illuminates the competition between nonlinearity and diffusion through many examples of shock formation, interaction, spreading and decay. Unfortunately, in nonlinear acoustics the Burgers equation has only limited applicability. Geometric spreading and material inhomogeneity both lead to a generalized Burgers equation in which ε^2 is replaced by a specified function $\varepsilon^2(Y)$. Although in some cases of interest these effects have been accommodated by the use of perturbation procedures (Crighton&Scott - 1979), no linearizing transformation has yet been found. It is therefore useful to obtain exact solutions of (1) without recourse to the Hopf-Cole transformation, in the hope that such solutions form the basis of a perturbation analysis that can accommodate modulation of ε^2 with Y.

In many problems of nonlinear acoustics the velocity u is periodic in x, and nonlinear effects are dominant near the source. The signal then develops a sawtoothprofile containing regularly spaced shocks. These shocks slowly increase in thickness until diffusion becomes important throughout the profile at large distances Y.

This paper is a continuation of [6; 7]. Numeric results are obtained via the Maple *PDE tools* package.

2. Stable smooth solutions

The Burgers (1) smooth stationary solutions are:

$$u(x,t) = c; \tag{3}$$

$$u(x,t) = -\varepsilon^2\alpha \tanh(\alpha x + c); \tag{4}$$

$$u(x,t) = -\varepsilon^2\alpha \coth(\alpha x + c); \tag{5}$$

$$u(x,t) = \varepsilon \cdot \alpha \tan(\frac{\alpha x + c}{\varepsilon}); \tag{6}$$

$$u(x,t) = \frac{\alpha \cdot \varepsilon^2}{\alpha x + c}. \tag{7}$$

Burgers equation on the whole line is known to possess travelling waves solutions with the saw-tool profiles (piecewise-smooth with periodical breaks of derivative). We show this property to have an analogue in a form of stationary, t-invariant saw-tool solutions.

Consider an IVBP for (1) of the form (2):

$$u(x,0) = f(x); \quad u(0,t) = A; \quad u(1,t) = B; \quad A, B \in R. \tag{8}$$

with smooth initial profile $f(x)$.

Taking the dissipation into the account it is naturally to presuppose that at $t \to \infty$ we get $u(x,t) \to y_{AB}(x)$ where $y_{AB}(x)$ is a unique smooth stationary solution corresponding to the ordinary differential problem $y'' - 2yy' = 0, \quad y(0) = A, \quad y(1) = B.$

Such solutions do exist and the first conjecture was that this limit does not depend on the initial profile $f(x)$.

Note that only bounded solutions (they are, incidentally, non-decreasing) are of interest if (1) is considered on the whole line $x \in R$. But on $x \in [a,b]$ anyone of the above list suits, providing the singularity is not on the interval.

3. Stability of invariant solutions

A solution of the Burgers equation

$$u_t = u_{xx} - \alpha u \cdot u_x \tag{9}$$

with zero boundary conditions

$$u(a,t) = u(b,t) = 0; \quad u(x,0)\big|_{[a,b]} = f(x) \tag{10}$$

monotonically tends to zero as $t \to \infty$ in L^2 norm since

$$\frac{\partial}{\partial t}\int_a^b u^2 dx = \int_a^b 2uu_t dx = 2\int_a^b u(u_{xx} - \alpha u\ u_x)dx = \tag{11}$$

$$2\int_a^b udu_x + \frac{-\alpha}{2}u^2\Bigg|_a^b = 2uu_x\big|_a^b - 2\int_a^b u_x^2 dx = -2\int_a^b u_x^2 dx \le 0 .$$

The greater u_x, the faster the convergence.

When the boundary conditions are non-zero but constant

$$u(x,0)\big|_{[a,b]} = f(x); \quad u(a,t) = f(a) = A; \quad u(b,t) = f(b) = B, \tag{12}$$

one may expect the solution to converge to the respective smooth stationary invariant solution, i.e. to $\mu(x)$:

$$\mu_{xx} - \alpha\mu \cdot \mu_x = 0; \quad \mu(a) = A; \quad \mu(b) = B . \tag{13}$$

Such a solution exists and is of one of the above listed forms depending on the combination of A and B . In fact, the situation is more complex.

In the case when dissipative effects are comparatively weak with respect to nonlinearity, other stationary solutions occur for the same IVBP. Namely these are the saw-tool solutions whose periods are $(b-a)/n$, $n \in N$.

Let us see how evolves the difference between u and the solution of (15). Denote $\nu(x,t) = u(x,t) - \mu(x)$, i.e. $u(t,x) = \nu(x,t) + \mu(x)$. Substituting the latter into (11) we get

$$u_t = (\nu(x,t) + \mu(x))_t = \nu(x,t)_t = u_{xx} - \alpha u \cdot u_x = \tag{14}$$

$$(\nu(x,t) + \mu(x))_{xx} - \alpha(\nu(x,t) + \mu(x))(\nu(x,t) + \mu(x))_x .$$

In the case $\alpha = 2$ it equals $\nu_{xx} - 2\nu\nu_x + [\mu_{xx} - 2\mu\mu_x] - 2\{\nu_x\mu + \nu\mu_x\}$. The expression in square brackets equals zero. So

$$\nu_t = \nu_{xx} - 2\nu\nu_x - 2(\nu\mu)_x . \tag{15}$$

Boundary conditions for ν are zero by definition. We evaluate the rate of ν by analogy with (13):

$$<\nu_t>\big|_{L^2} = \frac{\partial}{\partial t}\int_a^b \nu^2 dx = \int_a^b 2\nu\nu_t dx =$$

$$2\int_a^b \nu(\nu_{xx} - 2\nu\nu_x - 2(\nu\mu)_x)dx = 2\int_a^b \nu d\nu_x - \frac{4}{3}\nu^3\Bigg|_a^b - 4\int_a^b \nu d(\nu\mu) =$$

$$2\nu\nu_x\big|_a^b - 2\int_a^b \nu_x^2 dx - 4\nu(\nu\mu)\big|_a^b + 4\int_a^b \mu\nu\nu_x dx =$$

$$-2\int_a^b \nu_x^2 dx + 2\int_a^b \mu_x\nu^2 dx - 2\nu^2\mu\big|_a^b = -2\int_a^b \left(\nu_x^2 + \mu_x\nu^2\right)dx. \tag{16}$$

Thus the monotony of L^2-convergence is not automatically guaranteed; but it surely takes place, for instance in the case $\mu_x \ge 0$ (the case of the increasing initial profile, which agrees with the characteristics method).

It follows that $\mu_x > 0$ guarantees decay: if such conditions are satisfied, the deviation ν decays to zero. When the inequality $<\nu_t>\big|_{L^2} \ge 0$ fails (e.g., for decreasing initial profile) the difference ν doesn't necessarily tend to zero. Usually the evolution ends in catastrophe or decay, but it may stabilize half-way.

3.1. Decay

Here is an example of a decay towards a *decreasing* invariant solution. The initial profile is chosen in a vicinity of this solution and the right-hand side of (16) is negative. Consider the equation $u_t = \varepsilon^2 u_{xx} - 2uu_x$.

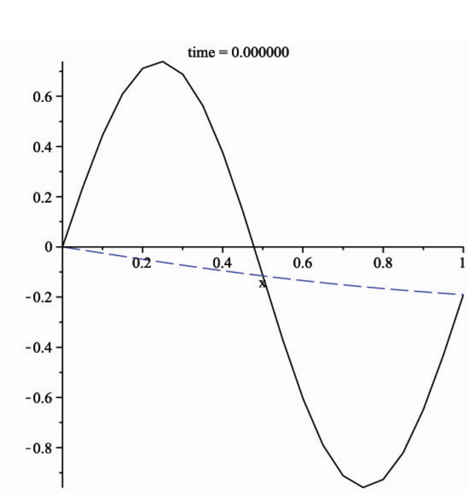

Fig. 1. Initial profile $-\varepsilon^2 \tanh(x) + 1.6\varepsilon \sin(2\pi x)$, $u(0,t) = 0,\ u(1,t) = -\varepsilon^2 \tanh(1)$. Asymptotic limit (dash line) is the invariant solution; n=1

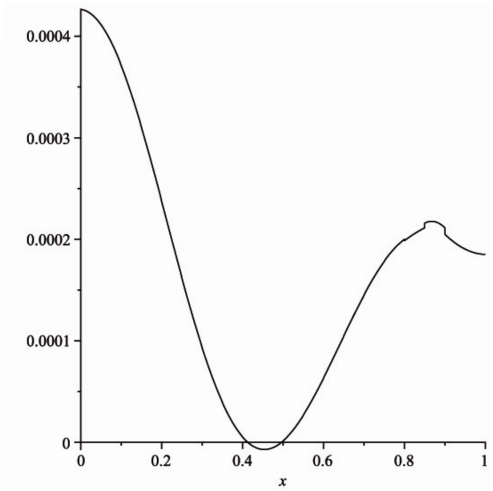

Fig. 2. The graph of the integrand $v_x^2 + \mu_x v^2$ in (16) for the solution on fig. 1 at $t = 2$

Choose IVBP: $u(x,0) = -\varepsilon^2 \tanh(x) + 1.6\varepsilon \sin(2\pi x),\ u(0,t) = 0,\ u(1,t) = -\varepsilon^2 \tanh(1)$; $\varepsilon = 0.05$. Here $\mu = -\varepsilon^2 \tanh(x)$ is a decreasing invariant solution, $v = 1.6\varepsilon \sin(2\pi x)$ — the perturbation. Asymptotics at $t \to \infty$ coincides with μ, see fig. 1. The dissipation reigns in and no catastrophe develops. The explanation can be seen in fig. 2 where the typical graph of integrand $v_x^2 + \mu_x v^2$ in (16) is given at $t = 2$; clearly $< v_t >|_{L^2} < 0$.

3.2. Catastrophes

As it is known, for a general quasilinear transport equation ($x \in R$)

$$w_t + f(w) w_x = 0 \tag{17}$$

the moment of gradient catastrophe can be defined as follows. Let $w = \varphi(x)$ be an initial profile. The solution of this problem may be given in a parametric form $w = \varphi(\xi),\ x = \xi + F(\xi)t$ where $F = f(\varphi(\xi))$. The characteristics of the form $x = \xi + F(\xi)t$ intersect in the case $\varphi'(\xi) < 0$ thus resulting in many-valued w (the tilting of a wave or a gradient catastrophe). If the inequality holds on a finite interval there exist a minimal value of time, t_c, when this problem arises. One may determine t_c by the formula $t_c = -1/F'(\xi_c)$ where $|F(\xi_c)| = \max|F'(\xi)|$ on the interval $[a,b]$. while $F'(\xi) < 0$.

We demonstrate this gradient catastrophe to be inherited by Burgers-like equations for some initial profiles, with modest dissipative effects added to a model (17); (cf [3; 2] dealing with a formation of dispersive shocks in a different class of extension of (17), namely Hamiltonian dispersive regularizations of (17) including KdV-likes and Kawahara equations).

In a complex environment of a finite interval combined with an added dissipation for the Burgers-like equation the catastrophe may be delayed or occur earlier, still the main features remain. We begin with the

Burgers equation $u_t = \varepsilon^2 u_{xx} - 2uu_x$, with IVBP $\{u(x,0) = \mathrm{sech}^2(x-1), u(0,t) = \mathrm{sech}^2(1), u(10,t) = \mathrm{sech}^2(9)\}$ and $\varepsilon = 0.02$. The initial peak moves to the right from the far left of the interval, so themoment t_c nearly coincides with that for a whole line. The ensuing multi-oscillating process results in a Heaviside-type break between boundary values at the right end of the interval, fig. 3, 4. Note that constants *are* invariant solutions.

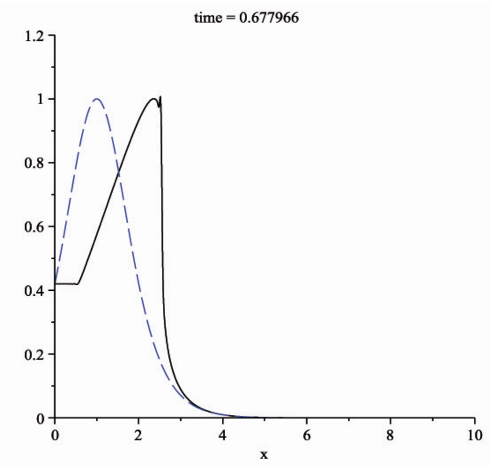

Fig. 3. Start of gradient catastrophe at $t_c \approx 0.67$. Dash line is the initial profile $\mathrm{sech}^2(x-1)$. n=1

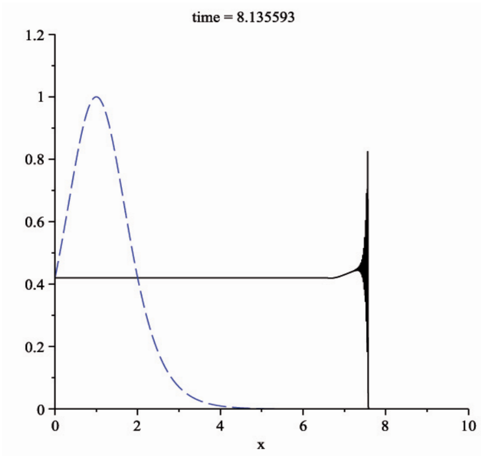

Fig. 4. Multi-oscillations move to a Heaviside type break $\tanh^2(1) - \tanh^2(9)$ at $x = 10$; $t \approx 8$

If we change IVBP of the previous problem for

$$\{u(x,0) = \mathrm{sech}^2(x-9),\ u(0,t) = \mathrm{sech}^2(9),\ u(10,t) = \mathrm{sech}^2(1)\},$$

the right end of the interval being nearer, the catastrophe begins earlier, at $t \approx 0.1$.

This is not a behavior specific for the sech^2 - type initial data. In yet one more example change the IVBP of the previous example for $u(x,0) = -0.01x^2 + 0.9,\ u(0,t) = 0.9,\ u(10,t) = -0.1$. The overall picture changes only slightly, fig. 7, though the catastrophe starts at $t = 3.9$, much later than $t = 1.9$ predicted by the characteristics method.

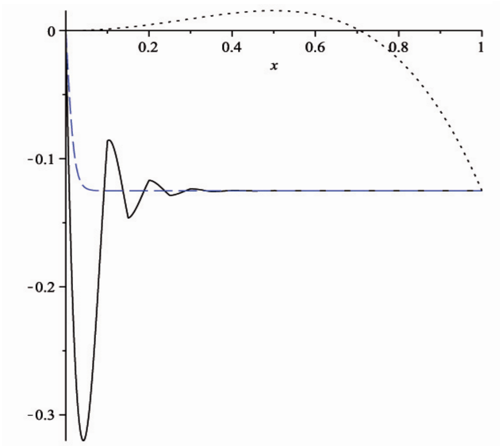

Fig. 5. Initial profile (dots line) $-\alpha\varepsilon^2 \tanh(\alpha)(2x^4 - x^2)$, wise- $u(0,t) = 0,\ u(1,t) = -\alpha\varepsilon^2 \tanh(\alpha)$. Asymptotic limit and the invariant solution (dash) $-\alpha\varepsilon^2 \tanh(x)$

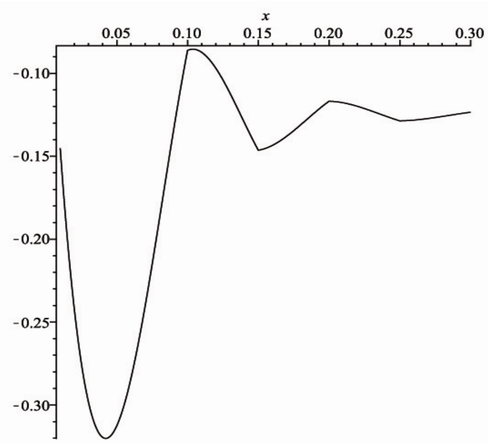

Fig. 6. Enlarged part of fig. 5. Piece- (solid) smooth difference $v(x)$; $t = 20$

3.3. Developing a stable saw-tool profiles

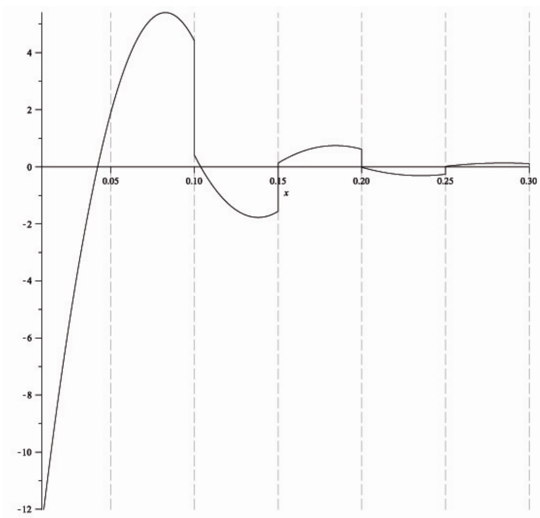

Fig. 7. The graph of the derivative v', $t=20$

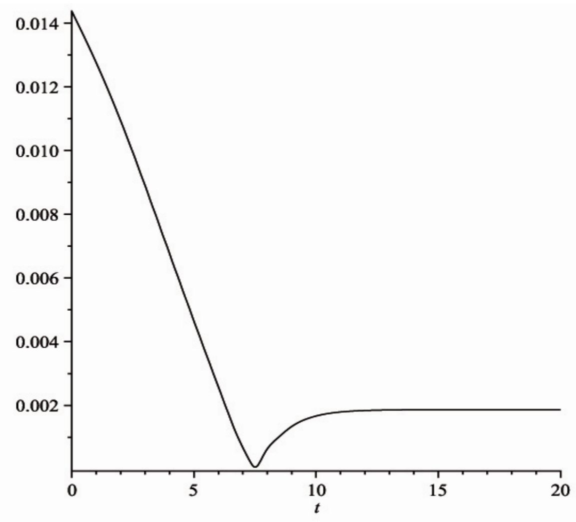

Fig. 8. L^2-estimate of difference between invariant $\mu(x)=-\alpha\varepsilon^2\tanh(\alpha x)$ and stabilizing multi-oscillating solution of fig. 5

In some cases the evolution of the initial profile results early and clearly *not* in an invariant solution from the list (3-7), see fig. 5 with IVBP

$$\{u(x,0)=-\alpha\cdot\varepsilon^2\tanh(\alpha)(2x^4-x^2),\ u(0,t)=0,\ u(1,t)=-\alpha\cdot\varepsilon^2\tanh(\alpha)\},\ \varepsilon=0.05;\alpha=50\ .$$

The stable graph is piecewise smooth. The effect is stable, as the final profile (solid line) here seems not to depend on wide variations of initial profile, provided boundary data is the same: identical asymptotics are obtained for $u(x,0)=-\alpha\varepsilon^2\tanh(\alpha)x$ or $-\alpha\cdot\varepsilon^2\tanh(\alpha)x^2$ (note that the invariant solution with the same boundary values is $\mu(x)=-\alpha\cdot\varepsilon^2\tanh(\alpha x)$).

The equation for the derivative $v=u'$ is $v_t=\varepsilon^2 v''-2v^2-2v'D^{-1}(v)$. The graph of derivative $u_x(x,20)$ is presented in fig. 7. Breaks form in a very early stage of evolution in vicinity of $t=0$.

The stabilization may be rather quick. The graph of L^2-estimate for the difference v, $<v>|_{L^2}=\int_0^1(u(s,t)-\mu(s))^2\,ds$ is presented in fig. 8.

Calculus of variations suggests to seek such a stationary point as an extremal of the functional (16)

$$\frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0}\int_a^b\left((v+\varepsilon h)_x^2-\mu_x(v+\varepsilon h)^2\right)dx=0.$$

It follows

$$v_{xx}+\mu_x v=0. \tag{18}$$

It is hard to compare the numeric extremal presented on fig. 8 to solutions of (18) by numeric methods. The obstacle is that the decreasing solutions of the Burgers equation are of the form $\mu(x)=-a\tanh(a\cdot x+b)$ and the potential of the linear problem (18), $\mu_x=\mathrm{sech}^2(ax+b)$, is numerically finite. As a result some of solutions of (18) are chaotic, at least numerically (e.g., the real part of complex solution of (18) may be both discontinuous and multi-oscillating).

The profile of solution shown in fig. 5 and 6 resemble an intermediate stage in development of a saw-tool solution for Burgers equation on the whole line (in the latter case a final stage line segments periodically altercate with breaks, as in a real saw, see [4; 5]; under Cole-Hopf transformation it corresponds to the solution of the heat equation which describes the spreading of a periodic array of point heat sources). One more specific feature is that the length of the segment $[a, b]$ is a multiple of the saw-teeth period, see fig. 7; on a face of it the period's length coincides with a spatial step of the numerical mesh, but the mesh points may act as sources of inherent shocks. This hints how to represent such solutions analytically. An exact description of such solutions is yet to be described in detail; it will be published elsewhere.

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ПИЛООБРАЗНЫЕ РЕШЕНИЯ УРАВНЕНИЯ БЮРГЕРСА НА ИНТЕРВАЛЕ

Самохин А.В., Дементьев Ю.И.

Изучается асимптотическое поведение решений уравнения Бюргерса на конечном интервале с заданными начальными и постоянными граничными условиями. Поскольку уравнение описывает движение в диссипативной среде, начальный профиль решения эволюционирует к стационарному (инвариантному по времени) решению с теми же граничными условиями. Однако к такому результату ведут три различных пути: начальный профиль может регулярно спускаться к гладкому инвариантному решению; или через дисперсионный шок и мульти-осцилляции развивается разрыв типа Хевисайда; или асимптотическим пределом оказывается пилообразное решение с периодическими разрывами производной.

Ключевые слова: уравнение Бюргерса, начально-граничная задача, градиентная катастрофа, пилообразные решения.

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