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A CONSTRUCTION OF DIFFEOMORPHISM EXTENSION AND ITS APPLICATIONS

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Let M be Riemannian manifold with boundary ∂M and a diffeomorphism f of ∂M. We consider the problem of the extension of f from the boundary ∂M into the manifold M to the volume-preserving diffeomorphism $\hat{f}$. The design of an explicit extension based on the representation theory is offered for the case of the sphere. We also extend the conformal and projective groups with the $n-1$-sphere into the n-ball. As a result, we construct examples of kinematic dynamo in the n-ball.

Keywords: Arnold's problems, Riemannian manifold, boundary, diffeomorphism extension, volume preserving diffeomorphism, kinematic dynamo.

Introduction

Let M be a compact Riemannian manifold with boundary ∂M and f be a diffeomorphism of ∂M. V.I. Arnold had formulated the following problem ([1], 1988-20): when there is an extension of f from ∂M into M such that the extension $\hat{f}$ is an element of the volume-preserving diffeomorphism group $\mathrm{SDiff}(M)$?

Here we consider the case of n-ball ($\mathbf{B}^n$) and propose an explicit construction of such an extension from its boundary, the $n-1$-sphere ($\mathbf{S}^{n-1}$), into $\mathbf{B}^n$. As a consequence we find new examples to the problem of kinematic dynamo in $\mathbf{B}^n$. These results were reported in the International conference "Analysis and singularities (Arnold-75)", [2].

1. Construction of the Extension

We consider diffeomorphisms of sphere which are isotopic to the identity. They form the group $\mathrm{Diff}_0(\mathbf{S}^{n-1}), n \geq 2$. Then for any diffeomorphism $f \in \mathrm{Diff}_0(\mathbf{S}^{n-1}$ we have (Thurston, [3])

$$f = \exp(v_1)\ldots\exp(v_k),$$

where v_i lies in $\mathrm{Vect}(\mathbf{S}^{n-1})$, the space of smooth vector fields on the $\mathbf{S}^{n-1}$, for $i = 1, \ldots, k$.

Let v be a smooth vector field on $\mathbf{S}^{n-1}$. We give a construction of its extension to a divergence-free vector field v in $\mathbf{B}^n$. Below we use the representation theory.

We decompose the space $\mathrm{Vect}(\mathbf{S}^{n-1})$ into irreducible $SO(n)$-modules. It is known (Kirillov, [4]), that there are two series of irreducible modules:

- The series of the divergence-free vector fields with highest weights $M_n + k\Lambda_n,\ k \geq 0$ for $n \neq 4$ and $M_n' + k\Lambda_n, M_n'' + k\Lambda_n, k \geq 0$ for $n = 4$:

$$S_1 = \mathrm{SVect}(\mathbf{S}^{n-1}) = \begin{cases} \sum\limits_{k\geq 0} P_{M_n + k\Lambda_n},\ n \neq 4; \\ \sum\limits_{k\geq 0} (P_{M_n' + k\Lambda_n} + P_{M_n'' + k\Lambda_n}),\ n = 4; \end{cases}$$

- The series of the gradient vector fields with highest weights $k\Lambda_n,\ k \geq 1$

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$$S_2 = \sum_{k\geq 1} P_{k\Lambda_n}.$$

Here M_n is the highest weight of the adjoint representation of $SO(n)$, $n \neq 4$ ($M_4^{'}, M_4^{''}$ for $SO(4)$); Λ_n is the highest weight of the standard representation of $SO(n)$ in $\mathbf{R}^n$. Thus, we have

$$\mathrm{Vect}(\mathbf{S}^{n-1}) = S_1 \oplus S_2.$$

We construct the extension for the irreducible modules of these series. In this case it is convenient to use the polynomial (in the sense of $\mathbf{R}^n$) notation of vector fields. Note that an irreducible $SO(n)$-module consists of spherical vector fields, in particular polynomial ones.

1.1. Divergence-free vector field series

Let us consider the vector field on $\mathbf{S}^n$

$$v_m = x_3^m (x_2, -x_1, 0, \ ..., 0), \ m \geq 0. \tag{1}$$

According to [5], v_m has non-null component in the space

$M_n + m\Lambda_n$ for $n \neq 4$; $M_n^{'} + m\Lambda_n$, $M_n^{''} + m\Lambda_n$ for $n = 4$.

Obviously, we have

$$\mathrm{div}_{\mathbf{S}^{n-1}} v_m = \mathrm{div}_{\mathbf{B}^n} v_m = 0 \cdot$$

It follows that the divergence-free extension of (1) is a vector field itself (i.e. $\hat{v}_m = v_m$). This property is invariant with respect to $SO(n)$ action on $\mathbf{S}^{n-1}$.

Thus we have

Proposition 1.1. There exists such a polynomial basis of divergence-free vector fields on $\mathbf{S}^n$ that its divergence-free extension into $\mathbf{B}^n$ arethe same vector fields.

Below we denote $\mathrm{div}_{\mathbf{B}^n} u$ by $\mathrm{div}\, u$.

1.2. Gradient series

In this case it is convenient to use harmonic polynomial form (Vilenkin, [6]). As it is well known [4] the homogenous harmonic polynomials of degree k form an irreducible space of highest weight $k\Lambda_n$, $k \geq 0$ in the function space on $\mathbf{R}^n$.

The corresponding space of their gradients on sphere constitutes a $SO(n)$-module with highest weight $k\Lambda_n$. One can offer the explicit formulas for these vector fields.

Consider the $SO(n)$-module with highest weight $k\Lambda_n$. It consists of the gradients of homogeneous harmonic polynomials p of the degree k. Let us take $u = \mathrm{grad}_{\mathbf{S}^{n-1}} p$. Then we have

$$u = (\frac{\partial p}{\partial x_1}, ..., \frac{\partial p}{\partial x_n}) - kp(x_1, ..., x_n). \tag{2}$$

Note that

$$\mathrm{div}\, u = \Delta p - k(n+k)p = -k(n+k)p.$$

Proposition 1.2. The divergence-free extension of (2) is given by the formula

$$\hat{u} = u + (\frac{n+k}{2})(x_1^2 + ... + x_n^2 - 1)(\frac{\partial p}{\partial x_1}, ..., \ \frac{\partial p}{\partial x_n}). \tag{3}$$

Proof.

We have

$$\mathrm{div}\, \hat{u} = -k(n+k)p + \frac{n+k}{2}(x_1^2 + ... + x_n^2 - 1)\Delta p + (n+k)\sum_{i=1}^{n} x_i \frac{\partial p}{\partial x_i} = 0.$$

2. Extension of actions of non-compact semi-simple Lie groups on sphere

In this section we construct volume-preserving extensions from a sphere into its interior ball for elements a non-compact simple Lie groupacting on the sphere.

There are known actions of Lie groups $SO(1,n)$ (the conformal group of a sphere) and $SL(n)$ (the projective group of a sphere) on $\mathbf{S}^{n-1}$ which do not automatically preserve the volume element of $\mathbf{S}^{n-1}$. Now we present the design of their extensions to volume-preserving diffeomorphisms of $\mathbf{B}^n$.

2.1. Consider the action of Lie group $SO(1,n)$ on $\mathbf{S}^{n-1}$.

Its Lie algebra consists of the vector fields

$$\{\operatorname{grad}_{\mathbf{S}^{n-1}} x_i, i=1,\dots,n\}; so(n).$$

For $n=3$, we choose

$$v=\operatorname{grad}_{\mathbf{S}^2} x_1=(1-x_1^2,-x_1x_2,-x_1x_3).$$

Using (3) we obtain that its divergence-free extension has the form

$$\hat{v}=(x_1^2+2x_2^2+2x_3^2-1,-x_1x_2,-x_1x_3). \tag{4}$$

Note that the vector field v does not have a singular point inside the ball $\mathbf{B}^3$. The vector field $\hat{v}$ singular points form the circle

$$x_1=0;\quad x_2^2+x_3^2=\frac{1}{2}.$$

For the vector field (4) it is immediately verified that

$$\operatorname{rot}\hat{v}=5(0,x_3,-x_2).$$

Thus we have

$$[\hat{v},\operatorname{rot}\hat{v}]=0.$$

Hence, the divergence-free extension of vector field $\operatorname{grad}_{\mathbf{S}^2} x_1$ is a vector field which commutes with its curl.

Take

$$w=\operatorname{grad}_{\mathbf{S}^2} x_2=(-x_1x_2,1-x_2^2,-x_2x_3).$$

Its divergence-free extension has the form

$$\hat{w}=(-x_1x_2,2x_1^2+x_2^2+2x_3^2-1,-x_2x_3). \tag{5}$$

Consider $h=(x_2,-x_1,0)$ as the vector field of the Lie algebra $so(n)$. Note that $\hat{h}=h$.

It is immediately verified that

$$[w+h,v]=w+h.$$

It follows

$$(\exp(v))^k(w+h)=e^k(w+h):$$

this is a property of the Lie group $SO(1,n)$.

Now we take

$$F=\exp(\hat{v}).$$

Note that F is an element of the volume-preserving diffeomorphism group $\operatorname{SDiff}(\mathbf{B}^n)$.

Thus, on the boundary of $\mathbf{B}^n$ (i.e. on $\mathbf{S}^{n-1}$) we obtain

$$F_*^k(\hat{w}+h)=e^k(\hat{w}+h),$$

For any integer k.

2.2. Consider the action of Lie groups $SL(n)$ on $\mathbf{S}^{n-1}$

Its Lie algebra consists of the vector fields from $so(n)$ and $\operatorname{grad}_{\mathbf{S}^{n-1}} x_ix_j, \operatorname{grad}_{\mathbf{S}^{n-1}}(x_i^2-x_j^2)$, $i,j=1,\dots,n, i\neq j$.

For $n=3$ we take $g=-x_1^2+x_2^2,\ u=\operatorname{grad}_{S^2} g$. Note that

$$u=(-2x_1,2x_2,0)-2g(x_1,x_2,x_3).$$

Its divergence-free extension has the form

$$\hat{u}=(5(x_1^2+x_2^2+x_3^2)-3)\,(-x_1,x_2,0)-2g(x_1,x_2,x_3). \tag{6}$$

Note that the vector field u has the set of singular points inside $\mathbf{B}^3$. This set coincides with the interval $x_1=x_2=0,\ -1<x_3<1$. The singular points of the vector field $\hat{u}$ inside $\mathbf{B}^3$ include this interval and also the two intersecting circles

$$x_1=\pm x_2;\ x_1^2+x_2^2+x_3^2=\frac{3}{5}.$$

Note that these extensions of the elements of conformal and projective group actions on the $\mathbf{S}^{n-1}$ generate infinite subgroups of the group $\mathrm{SDiff}(\mathbf{B}^n)$.

3. Extension of actions of non-commutative solvable Lie groups on sphere

Here we construct volume-preserving extensions for the non-commutative solvable Lie group actions on sphere into its interior ball.

Let us consider $\mathbf{S}^{2n-1},\ n\geq 2$. We take

$$h=(-x_2,x_1,-x_4,x_3,\dots,-x_{2n},x_{2n-1});\ v=p(x_1,x_2)\,(0,0,-x_4,x_3,\dots,-x_{2n},x_{2n-1}),$$

where p is a homogeneous harmonic polynomial of the degree k.

The vector fields h,v generate a finite-dimensional non-commutative solvable Lie algebra. Their divergence-free extensions from $\mathbf{S}^{n-1}$ into $\mathbf{B}^n$ are the same vector fields.

Example 3.1. Consider $\mathbf{S}^3$.

Let us take

$$h=(-x_2,x_1,-x_3,x_4);\ v=x_1x_2(0,0,-x_4,x_3);\ w=(x_1^2-x_2^2)\,(0,0,-x_4,x_3).$$

We have

$$[h,v]=w;\ [h,w]=-4v;\ [v,w]=0.$$

Thus, we obtain the 3-dimensional non-commutative solvable Lie algebra of divergence-free vector fields on $\mathbf{S}^3$. Theirs divergence-free extensions on $\mathbf{B}^4$ are the same vector fields.

Note that these extensions of the elements of non-commutative solvable group actions on the $\mathbf{S}^{2n-1}$ generate a finite Lie group as subgroups of $\mathrm{SDiff}(\mathbf{B}^{2n})$.

4. Application to the problem of kinematic dynamo

Let us consider the vector field $\hat{u}$ (6). Note that $\hat{u}$ has a singular point at the origin. Linearized vector field $\frac{\partial \hat{u}}{\partial x}$ at the origin has the form

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and its eigenvalues are $\{3,-3,0\}$.

According to Arnold-Khes in theorem [7, p. 275-276], $\hat{u}$ is an example of nondissipative kinematic dynamo of the rate $L_{1+\tau}$ for any $\tau>0$. Direction for implementing dynamo effect at the origin is given by the eigenvector (1, 0, 0).

Divergence-free vector field with the focus at the origin may be obtained using an extension of the conformal group. Here again, we take the vector field $\hat{v}$, (4). Note that $\hat{v}(0) = (-1, 0, 0)$. Consequently, the vector field $\hat{v}$ can be considered as the effect of the magnetic field in the kinematic dynamo.

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КОНСТРУКЦИЯ ПРОДОЛЖЕНИЯ ДИФФЕОМОРФИЗМА И ЕГО ПРИЛОЖЕНИЯ

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Пусть M есть риманово многообразие с границей ∂M, а f есть диффеоморфизм ∂M. Рассмотрим задачу продолжения f с границы ∂M внутрь многообразия M до сохраняющего объём диффеоморфизма $\hat{f}$. В статье для случая сферы предложена явная конструкция такого продолжения, основывающаяся на теории представлений. Мы также рассматриваем продолжения действия конформной и проективной групп с $n-1$-сферы на n-мерный шар. В результате получены примеры кинематического динамов n-мерный шаре.

Ключевые слова: проблема Арнольда, риманово многообразие, граница, продолжение диффеоморфизма, диффеоморфизм сохраняющий объём, кинематическое динамо.

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Lukatsky Alexander Mikhajlovich, born in 1949, graduated from the Moscow State University named after Lomonosov (1972), Sci.D. in Physical and Mathematical Sciences, Senior Research Fellow of the Institute of Energy Research Institute Russian Academy of Sciences (ERIRAS), author of 96 scientific papers, research interests – infinite groups application to equations of mathematical physics.