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AFFINE CONNECTION ADJOINED TO WEB W(1, N, 1)

A.A. DUYUNOVA

The article is presented by professor, Science Doctor in Technics A.V. Samokhin

The 3-web $W(1,n,1)$ and the system of differential equations are adjoined with an associated affine torsion-free connection (the so called canonical connection of an ordinary differential equations' system). Components of the torsion tensor of this connection are expressed through the functions defining the system of differential equations. A general form of the system of differential equations with the zero torsion tensor is obtained.

Key words: multidimensional three-web, system of ordinary differential equations, affine connection.

Introduction

The paper is a continuation of [3], where we set the correspondence between the systems of ordinary differential equations and 3-webs $W(1,n,1)$ formed by foliations of dimension $1,n,1$ on the manifold of dimension $n+1$. In [3] we had found structural equations for the $W(1,n,1)$-web and its principal tensors. In the present paper their components and differential forms contained in the structural equations are expressed through functions that define the system of differential equations; we also clarify the geometrical content of reducing the first structural tensor to zero (see Theorem 1 below).

Let M be a smooth manifold of dimension $n+1$. Consider a 3-web on it given by families λ_1 and λ_3 of curves and family λ_2 of hypersurfaces. Denote by $T_p(M)$ a space tangent to manifold M at point p, and by $T_p(F_\alpha)$, $\alpha=1,2,3$, spaces tangent to foliations F_α of web W at this point. At the same point p we consider the manifold $R(W)$ of adapted frames e_a, $a,b,\ldots=1,2,\ldots,n+1$, whose first n vectors lie in $T_p(F_2)$, e_{n+1} lies in $T_p(F_1)$, and vector e_n-e_{n+1} - in $T_p(F_3)$. It is shown in [3] that in the above described frame of the family λ_α webs $W(1,n,1)$ are given by the following Pfaffian equations:

$$\lambda_1:\omega^u=0,\omega^n=0;$$
$$\lambda_2:\omega^{n+1}=0;$$
$$\lambda_3:\omega^u=0,\omega^n+\omega^{n+1}=0,$$

where $\{\omega^u,\omega^n,\omega^{n+1}\}$ is a dual coframe, and $u,v,\ldots=1,2,\ldots,n-1$. The group of admissible transformations preserving the form of the equations defines G-structure on the manifold of 3-web $W(1,n,1)$. The introduced forms satisfy the following structural equations:

$$\begin{aligned} d\omega^u&=\omega^v\wedge\omega_v^u+\mu^u\omega^n\wedge\omega^{n+1};\\ d\omega^n&=\omega^u\wedge\omega_u^n+\omega^n\wedge\omega_n^n;\\ d\omega^{n+1}&=\omega^{n+1}\wedge\omega_n^n, \end{aligned}\tag{1}$$

and quantities μ^u form a tensor on G-structure. It is called *the first structural tensor* of 3-web $W(1,n,1)$. The differential extension of (1) leads to the equations:

$$d\omega_v^u = \omega_v^w \wedge \omega_w^u + \mu^u \omega_v^n \wedge \omega^{n+1} + k_v^u \omega^n \wedge \omega^{n+1} - \omega^w \wedge \omega_{vw}^u;$$
$$d\omega_u^n = \omega_u^v \wedge \omega_v^n + \omega_u^n \wedge \omega_n^n + t_u \omega^n \wedge \omega^{n+1} - \omega^v \wedge \omega_{uv}^n;$$
$$d\omega_n^n = \mu^u \omega^{n+1} \wedge \omega_u^n + t_u \omega^u \wedge \omega^{n+1} + t_n \omega^n \wedge \omega^{n+1};$$
$$d\mu^u = -\mu^v \omega_v^u + 2\mu^u \omega_n^n + k_v^u \omega^v + k_n^u \omega^n + k_{n+1}^u \omega^{n+1}; \qquad (2)$$

$$d\omega_{vw}^u + \omega_s^u \wedge \omega_{vw}^s - \omega_v^s \wedge \omega_{sw}^u - \omega_w^s \wedge \omega_{vs}^u - \mu^u \omega_{vw}^n \wedge \omega^{n+1} =$$
$$= -k_w^u \omega_v^n \wedge \omega^{n+1} - k_v^u \omega_w^n \wedge \omega^{n+1} - h_{vw}^u \omega^n \wedge \omega^{n+1} + \omega_{vws}^u \wedge \omega^s;$$
$$d\omega_{uv}^n - \omega_{uv}^w \wedge \omega_w^n - \omega_u^w \wedge \omega_{wv}^n - \omega_{uv}^n \wedge \omega_n^n + \omega_{uw}^n \wedge \omega_v^w =$$
$$= -t_v \omega_u^n \wedge \omega^{n+1} - t_u \omega_v^n \wedge \omega^{n+1} - m_{uv} \omega^n \wedge \omega^{n+1} + \omega_{uvw}^n \wedge \omega^w;$$
$$dt_u - t_v \omega_u^v - t_u \omega_n^n - t_n \omega_u^n + k_u^v \omega_v^n = m_{uv} \omega^v + m_{un} \omega^n + m_{u\,n+1} \omega^{n+1} + \mu^v \omega_{vu}^n; \qquad (3)$$
$$dt_n - 2t_n \omega_n^n + k_n^u \omega_u^n = m_{un} \omega^u + m_{nn} \omega^n + m_{n\,n+1} \omega^{n+1};$$
$$dk_v^u + k_v^w \omega_w^u - k_w^u \omega_v^w - k_n^u \omega_v^n - 2k_v^u \omega_n^n = h_{vw}^u \omega^w + h_{vn}^u \omega^n + h_{v\,n+1}^u \omega^{n+1} + \mu^w \omega_{wv}^u;$$
$$dk_n^u + k_n^v \omega_v^u - 3k_n^u \omega_n^n = h_{vn}^u \omega^v + h_{nn}^u \omega^n + h_{n\,n+1}^u \omega^{n+1};$$
$$dk_{n+1}^u + k_{n+1}^v \omega_v^u - 3k_{n+1}^u \omega_n^n = 3\mu^u \mu^v \omega_v^n + \left(h_{v\,n+1}^u - 2\mu^u t_v\right)\omega^v + \left(h_{n\,n+1}^u - 2\mu^u t_n\right)\omega^n + h_{n+1\,n+1}^u \omega^{n+1},$$

and the following relations take place:

$$h_{vw}^u = h_{wv}^u; \qquad m_{uv} = m_{vu};$$
$$\omega_{uv}^n = \omega_{vu}^n; \qquad \omega_{vw}^u = \omega_{wv}^u; \qquad \omega_{uvw}^n = \omega_{uwv}^n; \qquad \omega_{vws}^u = \omega_{vsw}^u.$$

The set of quantities $\{t_u, t_n, k_v^u, k_n^u, k_{n+1}^u\}$ forms a tensor on $\breve{G}$-structure, where $\breve{G}$ is a subgroup of the extended group G. It is called *the second structural tensor* of 3-web $W(1,n,1)$.

By [3], the system of differential equations

$$\frac{dx^i}{dt} = f^i\left(t, x^1, \ldots, x^n\right) \quad , \quad \left(i, j, \ldots = 1, \ldots, n\right) \qquad (4)$$

is related with a 3-web $W(1,n,1)$ on manifold of variables x^i, t consisting of families λ_α, where

$$\lambda_1 : x^i = const; \qquad \lambda_2 : t = const; \qquad \lambda_3 : F^i(t, x^j) = c^i = const$$

(the latter family consists of integral curves of the system (4)). Components of the first and second structural tensors of the web are expressed through the functions f^i defining the system of ordinary differential equations:

$$\mu^u = f^u \frac{\partial f^n}{\partial t} - f^n \frac{\partial f^u}{\partial t};$$

$$t_u = \frac{1}{\left(f^n\right)^3} \frac{\partial f^n}{\partial x^u} \frac{\partial f^n}{\partial t} - \frac{1}{\left(f^n\right)^2} \frac{\partial^2 f^n}{\partial x^u \partial t};$$

$$t_n = \frac{1}{f^n}\left(\frac{\partial f^n}{\partial x^n}\frac{\partial f^n}{\partial t} + \frac{\partial f^n}{\partial x^u}\frac{\partial f^u}{\partial t} - f^u \frac{\partial^2 f^n}{\partial x^u \partial t}\right) - \frac{\partial^2 f^n}{\partial x^n \partial t};$$

$$k_v^u = \frac{1}{f^n}\left(\frac{\partial f^u}{\partial x^v}\frac{\partial f^n}{\partial t} + f^u \frac{\partial^2 f^n}{\partial x^v \partial t} - \frac{\partial f^n}{\partial x^v}\frac{\partial f^u}{\partial t} - f^n \frac{\partial^2 f^u}{\partial x^v \partial t} + \frac{f^w}{f^n}\frac{\partial f^n}{\partial t}\frac{\partial f^n}{\partial x^w}\delta_v^u - \frac{\partial f^w}{\partial t}\frac{\partial f^n}{\partial x^w}\delta_v^u\right);$$

$$k_n^u = f^u f^v \frac{\partial^2 f^n}{\partial x^v \partial t} - f^v f^n \frac{\partial^2 f^u}{\partial x^v \partial t} + f^u f^n \frac{\partial^2 f^n}{\partial x^n \partial t} - (f^n)^2 \frac{\partial^2 f^u}{\partial x^n \partial t} +$$

$$+ f^n \frac{\partial f^u}{\partial x^n} \frac{\partial f^n}{\partial t} + f^n \frac{\partial f^v}{\partial t} \frac{\partial f^u}{\partial x^v} - f^u \frac{\partial f^n}{\partial t} \frac{\partial f^n}{\partial x^n} - f^u \frac{\partial f^v}{\partial t} \frac{\partial f^n}{\partial x^v};$$

$$k_{n+1}^u = f^n \frac{\partial^2 f^u}{\partial t^2} - f^u \frac{\partial^2 f^n}{\partial t^2} + 3 \frac{f^u}{f^n} \left(\frac{\partial f^n}{\partial t} \right)^2 - 3 \frac{\partial f^u}{\partial t} \frac{\partial f^n}{\partial t}.$$

The forms contained in the structural equations (1), (2) can be expressed through dx^i and f^i. In particular, these formulas allow proving the following statement.

Theorem 1. A system of ordinary differential equations is autonomous iff μ^u and ω_n^n are equal to zero.

1. Affine connection

It is known (see, for example, [4]) that on manifold M with an affine connection, the Pfaffian forms θ^a, θ_b^a, $a,b,\ldots = 1,2,\ldots,n+1$ that define the connection satisfy equations of the form:

$$d\theta^a = \theta^b \wedge \theta_b^a + R_{bc}^a \theta^b \wedge \theta^c; \qquad (5)$$
$$d\theta_b^a = \theta_b^c \wedge \theta_c^a + R_{bcd}^a \theta^c \wedge \theta^d,$$

where R_{bc}^a, R_{bcd}^a are the torsion and curvature tensors of the connection respectively, and quadratic forms $\Omega^a = R_{bc}^a \theta^b \wedge \theta^c$ and $\Omega_b^a = R_{bcd}^a \theta^c \wedge \theta^d$ are the torsion and curvature forms of the connection. We can see from equations (1) and (2) that, generally, they are not structural equations of an affine connection.

Theorem 2. Equations (1) and (2) on manifold M define an affine connection without torsion iff the forms ω_u^n, ω_{uv}^n and ω_{vw}^u are principal, i.e., they are expressed through the basic forms ω^u, *omega*^{n} and ω^{n+1}.

Sufficiency. For the equations (1) and (2) we put

$$\theta^a \equiv (\omega^u, \omega^n, \omega^{n+1});$$

$$\theta_b^a = \begin{pmatrix} \theta_v^u & \theta_n^u & \theta_{n+1}^u \\ \theta_u^n & \theta_n^n & \theta_{n+1}^n \\ \theta_u^{n+1} & \theta_n^{n+1} & \theta_{n+1}^{n+1} \end{pmatrix} \equiv \begin{pmatrix} \omega_v^u & \mu^u \omega^{n+1} & 0 \\ \omega_u^n & \omega_n^n & 0 \\ 0 & 0 & \omega_n^n \end{pmatrix}, \qquad (6)$$

and

$$\omega_u^n = \lambda_{uv}^n \omega^v + \lambda_{un}^n \omega^n + \lambda_{u\,n+1}^n \omega^{n+1};$$
$$\omega_{uv}^n = \lambda_{uvw}^n \omega^w + \lambda_{uvn}^n \omega^n + \lambda_{uv\,n+1}^n \omega^{n+1}; \qquad (7)$$
$$\omega_{vw}^u = \lambda_{vws}^u \omega^s + \lambda_{vwn}^u \omega^n + \lambda_{vw\,n+1}^u \omega^{n+1}.$$

Then the connection forms obtained this way satisfy the equations (5). Indeed, the forms θ_v^u, θ_u^n, θ_n^n and θ_{n+1}^{n+1} become structural equations of web (2); and for the forms θ_n^u by (6) and (2) we get:

$$d\theta_n^u - \theta_n^v \wedge \theta_v^u - \theta_n^n \wedge \theta_n^u - \theta_n^{n+1} \wedge \theta_{n+1}^u =$$
$$= d\mu^u \wedge \omega^{n+1} + \mu^u d\omega^{n+1} - \mu^v \omega^{n+1} \wedge \omega_v^u - \mu^u \omega_n^n \wedge \omega^{n+1} =$$
$$= \left(-\mu^v \omega_v^u + 2\mu^u \omega_n^n + k_v^u \omega^v + k_n^u \omega^n + k_{n+1}^u \omega^{n+1} \right) \wedge \omega^{n+1} + \mu^u \omega^{n+1} \wedge \omega_n^n -$$
$$- \mu^v \omega^{n+1} \wedge \omega_v^u - \mu^u \omega_n^n \wedge \omega^{n+1} = k_v^u \omega^v \wedge \omega^{n+1} + k_n^u \omega^n \wedge \omega^{n+1}.$$

The right-hand side of the latter equation contains only combinations of basic forms, which proves the statement.

Necessity. Let structural equations (1), (2) set affine connection without torsion. Comparing equations (1) and equations (5) for the connection we see that components of connection forms may be chosen only in the form (6). Since equations (2) must have the form (5) then firstly, the forms ω^n_{uv} and ω^u_{vw} must be principal, i.e., relations (7) must hold; secondly, the last two equations of (5) have the form:

$$\theta^n_n = \theta^u_n \wedge \theta^n_u + \theta^{n+1}_n \wedge \theta^n_{n+1} + R^n_{nuv}\,\theta^u \wedge \theta^v + 2R^n_{nun}\,\theta^u \wedge \theta^n + 2R^n_{nu\,n+1}\,\theta^u \wedge \theta^{n+1} + 2R^n_{nn\,n+1}\,\theta^n \wedge \theta^{n+1};$$

$$d\theta^{n+1}_{n+1} = \theta^u_{n+1} \wedge \theta^{n+1}_u + \theta^n_{n+1} \wedge \theta^{n+1}_n + R^{n+1}_{n+1uv}\,\theta^u \wedge \theta^v + 2R^{n+1}_{n+1un}\,\theta^u \wedge \theta^n + 2R^{n+1}_{n+1u\,n+1}\,\theta^u \wedge \theta^{n+1} + 2R^{n+1}_{n+1n\,n+1}\,\theta^n \wedge \theta^{n+1}.$$

In view of (6) the right-hand sides of these equations must coincide. But the right part of the latter equation contains only basic forms since the first two addends are equal to zero (see (6)). Therefore the form $\theta^u_n \wedge \theta^n_u$ contained in the right-hand side of the first equation must be expressed through basic forms. Since the forms θ^u_n are principal (see (6)) it follows that the forms θ^n_u are expressed through basic forms, i.e., the relations (7) hold.

Following [2] we call connections satisfying the hypotheses of Theorem 2 by *connections associated* with 3-web $W(1,n,1)$ and denote them by Γ.

Hereafter, speaking of affine connections on 3-web $W(1,n,1)$ we will mean associated connections.

Using equations (1), (2) and (7), we find nonzero components of curvature tensor of the connection Γ:

$$R^u_{vws} = -\lambda^u_{v[ws]};\quad R^u_{vwn} = -\frac{1}{2}\lambda^u_{vwn};\quad R^u_{vw\,n+1} = -\frac{1}{2}\lambda^u_{vw\,n+1}4;\quad R^u_{vn\,n+1} = \frac{1}{2}k^u_v;$$

$$R^u_{nv\,n+1} = \frac{1}{2}k^u_v;\quad R^u_{nn\,n+1} = \frac{1}{2}k^u_n;$$

$$R^n_{uvw} = -\lambda^n_{u[vw]};\quad R^n_{uvn} = -\frac{1}{2}\lambda^n_{uvn};\quad R^n_{uv\,n+1} = -\frac{1}{2}\lambda^n_{uv\,n+1};\quad R^n_{un\,n+1} = \frac{1}{2}t_u;$$

$$R^n_{nu\,n+1} = \frac{1}{2}t_u;\quad R^n_{nn\,n+1} = \frac{1}{2}t_n;$$

$$R^{n+1}_{n+1u\,n+1} = \frac{1}{2}\left(t_u - \mu^v\lambda^n_{vu}\right);\quad R^{n+1}_{n+1n\,n+1} = \frac{1}{2}\left(t_n - \mu^u\lambda^n_{un}\right).$$

Theorem 3. For all affine connections Γ associated with 3-web $W(1,n,1)$, and only for such connections, foliations of the web are completely geodesic.

Parallel transfer of the vector $\xi = \{\xi^a\}$ in affine connection is defined by the equations

$$d\xi^a + \xi^b\theta^a_b = 0,$$

while equations of geodesic lines of the connection have the form

$$d\theta^a + \theta^b\theta^a_b = \theta\theta^a, \qquad (8)$$

where d is the symbol of the ordinary (not exterior) differentiation, and θ is a Pfaffian form.

Let us set on manifold M^{n+1} carrying 3-web $W(1,n,1)$ an affine connection associated with the web. Since the form matrix of the such a connection has the form (6), the equations of geodesic lines (8) are of the form:

$$d\omega^u + \omega^v \omega_v^u + \mu^u \omega^n \omega^{n+1} = \theta\omega^u;$$
$$d\omega^n + \omega^u \omega_u^n + \omega^n \omega_n^n = \theta\omega^n; \qquad (9)$$
$$d\omega^{n+1} + \omega^{n+1}\omega_n^n = \theta\omega^{n+1}.$$

It follows from (9) that the lines of web $W(1,n,1)$ of family λ_1 defined by equations $\omega^u = 0$, $\omega^n = 0$ are geodesic with respect to the connection Γ. The surfaces of the second family λ_2 defined by the equation $\omega^{n+1} = 0$ are completely geodesic since the last equation of (9) is identically satisfied, while the other two define geodesic lines on the surfaces. By analogy, for the lines of the web in the third family λ_3 defined by equations $\omega^u = 0$ and $\omega^n + \omega^{n+1} = 0$ the first equation of (9) is satisfied identically, while the second and third equations are equivalent. We find the form θ from this unique equation.

Inversely, consider on a manifold M^{n+1} carrying the 3-web $W(1,n,1)$ an affine connection Γ without torsion with the forms θ_b^a. The geodesics equations for the 3-web have the common form:

$$d\omega^u + \omega^v\theta_v^u + \omega^n\theta_n^u + \omega^{n+1}\theta_{n+1}^u = \theta\omega^u;$$
$$d\omega^n + \omega^u\theta_u^n + \omega^n\theta_n^n + \omega^{n+1}\theta_{n+1}^n = \theta\omega^n; \qquad (10)$$
$$d\omega^{n+1} + \omega^u\theta_u^{n+1} + \omega^n\theta_n^{n+1} + \omega^{n+1}\theta_{n+1}^{n+1} = \theta\omega^{n+1}.$$

Consider the web $W(1,n,1)$. Let the lines of the first foliation λ_1 defined by equations $\omega^u = 0$ and $\omega^n = 0$ be geodesic for this connection. Then from (10) we get $\theta_{n+1}^u = 0$ and $\theta_{n+1}^n = 0$ modulo forms ω^u, ω^n. If the second foliation λ_2 defined by the equation $\omega^{n+1} = 0$ is completely geodesic, from (10) it follows $\omega^u\theta_u^{n+1} + \omega^n\theta_n^{n+1} = 0$. Since geodesic lines on the completely geodesic surface $\omega^{n+1} = 0$ may pass in any direction, the last equality is satisfied identically with respect to ω^u and ω^n. It follows then $\theta_u^{n+1} = 0$ and $\theta_n^{n+1} = 0$ modulo form ω^{n+1}. Finally, if the lines of the third foliations of 3-web W defined by equations $\omega^u = 0$ and $\omega^n + \omega^{n+1} = 0$ are geodesic, from the same equations (10) we obtain $\theta_n^n = \theta_{n+1}^{n+1}$ and $\theta_n^u = 0$ modulo forms ω^u and $\omega^n + \omega^{n+1}$.

Thus, for any connection Γ such that the leaves of web $W(1,n,1)$ are completely geodesic the following relations hold:

$$\theta_{n+1}^u = \chi_{n+1v}^u\omega^v + \chi_{n+1n}^u\omega^n;$$
$$\theta_{n+1}^n = \chi_{n+1v}^n\omega^v + \chi_{n+1n}^n\omega^n;$$
$$\theta_u^{n+1} = \chi_{un+1}^{n+1}\omega^{n+1};$$
$$\theta_n^{n+1} = \chi_{nn+1}^{n+1}\omega^{n+1}; \qquad (11)$$
$$\theta_{n+1}^{n+1} = \theta_n^n + \chi_{n+1v}^{n+1}\omega^v + \chi_{n+1n}^{n+1}(\omega^n + \omega^{n+1});$$
$$\theta_n^u = \chi_{nv}^u\omega^v + \chi_{nn}^u\left(\omega^n + \omega^{n+1}\right).$$

We see that the choice of connection Γ is of great arbitrariness. In this sense we can speak about a family of connections Γ.

Assuming (11) in the first set of equations (5), we have:

$$d\omega^u = \omega^v \wedge \theta_v^u + \omega^n \wedge (\chi_{nv}^u\omega^v + \chi_{nn}^u\omega^{n+1}) + \omega^{n+1} \wedge (\chi_{n+1v}^u\omega^v + \chi_{n+1n}^u\omega^n);$$
$$d\omega^n = \omega^u \wedge \theta_u^n + \omega^n \wedge \theta_n^n + \omega^{n+1} \wedge (\chi_{n+1v}^n\omega^v + \chi_{n+1n}^n\omega^n);$$
$$d\omega^{n+1} = \chi_{un+1}^{n+1}\omega^u \wedge \omega^{n+1} + \chi_{nn+1}^{n+1}\omega^n \wedge \omega^{n+1} + \omega^{n+1} \wedge (\theta_n^n + \chi_{n+1v}^{n+1}\omega^v + \chi_{n+1n}^{n+1}\omega^n);$$

or

$$d\omega^u = \omega^v \wedge \left(\theta^u_v - \chi^u_{nv}\omega^n - \chi^u_{n+1v}\omega^{n+1}\right) + \omega^n \wedge \omega^{n+1}(\chi^u_{nn} - \chi^u_{n+1n});$$

$$d\omega^n = \omega^u \wedge \left(\theta^n_u + \left(\chi^{n+1}_{n+1u} + \chi^{n+1}_{u\,n+1}\right)\omega^n - \chi^n_{n+1u}\omega^{n+1}\right) +$$

$$+\omega^n \wedge \left(\theta^n_n + \left(\chi^{n+1}_{n+1u} - \chi^{n+1}_{u\,n+1}\right)\omega^u + \left(\chi^{n+1}_{n+1n} - \chi^{n+1}_{n\,n+1}\right)\omega^n - \chi^n_{n+1n}\omega^{n+1}\right)\Big);$$

$$d\omega^{n+1} = \omega^{n+1} \wedge \left(\theta^n_n + \left(\chi^{n+1}_{n+1u} - \chi^{n+1}_{u\,n+1}\right)\omega^u + \left(\chi^{n+1}_{n+1n} - \chi^{n+1}_{n\,n+1}\right)\omega^n - \chi^n_{n+1n}\omega^{n+1}.$$

Denoting

$$\tilde{\theta}^u_v = \theta^u_v - \chi^u_{nv}\omega^n - \chi^u_{n+1v}\omega^{n+1};$$

$$\tilde{\mu}^u = \chi^u_{nn} - \chi^u_{n+1n};$$

$$\tilde{\theta}^n_u = \theta^n_u + \left(\chi^{n+1}_{n+1u} + \chi^{n+1}_{u\,n+1}\right)\omega^n - \chi^n_{n+1u}\omega^{n+1};$$

$$\tilde{\theta}^n_n = \theta^n_n + \left(\chi^{n+1}_{n+1u} - \chi^{n+1}_{u\,n+1}\right)\omega^u + \left(\chi^{n+1}_{n+1n} - \chi^{n+1}_{n\,n+1}\right)\omega^n - \chi^n_{n+1n}\omega^{n+1};$$

we get

$$d\omega^u = \omega^v \wedge \tilde{\theta}^u_v + \tilde{\mu}^u \omega^n \wedge \omega^{n+1};$$

$$d\omega^n = \omega^u \wedge \tilde{\theta}^n_u + \omega^n \wedge \tilde{\theta}^n_n;$$

$$d\omega^{n+1} = \omega^{n+1} \wedge \tilde{\theta}^n_n.$$

The latter equations coincide with equations (1), if we put $\omega^u_v = \tilde{\theta}^u_v$, $\mu^u = \tilde{\mu}^u$, $\omega^n_u = \tilde{\theta}^n_u$, $\omega^n_n = \tilde{\theta}^n_n$. Note that Theorems 2 and 3 were formulated in [2]. However, their proofs were not given.

2. System of ordinary differential equations with zero curvature tensor

By [3], the forms ω^n_u, ω^n_{uv} and ω^u_{vw} are expressed through functions f^i defining the system of ordinary differential equations:

$$\omega^n_u = -\frac{1}{\left(f^n\right)^3}\frac{\partial f^n}{\partial x^u}dx^n; \tag{12}$$

$$\omega^n_{uv} = \frac{1}{\left(f^n\right)^4}\left(\frac{\partial^2 f^n}{\partial x^v \partial x^u} - \frac{4}{f^n}\frac{\partial f^n}{\partial x^u}\frac{\partial f^n}{\partial x^v}\right)dx^n; \tag{13}$$

$$\omega^u_{vw} = \frac{1}{\left(f^n\right)^3}\left(\frac{2}{f^n}\frac{\partial f^n}{\partial x^w}\frac{\partial f^n}{\partial x^v} - \frac{\partial^2 f^n}{\partial x^w \partial x^v}\right)\omega^u +$$

$$+\frac{1}{\left(f^n\right)^2}\left(-\frac{1}{f^n}\frac{\partial f^n}{\partial x^v}\frac{\partial f^n}{\partial t}\delta^u_w - \frac{1}{f^n}\frac{\partial f^n}{\partial x^w}\frac{\partial f^n}{\partial t}\delta^u_v + \frac{\partial^2 f^n}{\partial t \partial x^v}\delta^u_w + \frac{\partial^2 f^n}{\partial x^w \partial t}\delta^u_v\right)dt +$$

$$+\frac{1}{\left(f^n\right)^3}\left(-\frac{\partial f^n}{\partial x^w}\frac{\partial f^u}{\partial x^v} - \frac{\partial f^u}{\partial x^w}\frac{\partial f^n}{\partial x^v} - 2\frac{f^s}{f^n}\frac{\partial f^n}{\partial x^w}\frac{\partial f^n}{\partial x^s}\delta^u_v - 2\frac{f^s}{f^n}\frac{\partial f^n}{\partial x^s}\frac{\partial f^n}{\partial x^v}\delta^u_w - \frac{\partial f^n}{\partial x^w}\frac{\partial f^n}{\partial x^n}\delta^u_v + \right.$$

$$+\frac{\partial f^n}{\partial x^s}\frac{\partial f^s}{\partial x^w}\delta^u_v + \frac{\partial f^n}{\partial x^s}\frac{\partial f^s}{\partial x^v}\delta^u_w + f^s\frac{\partial^2 f^n}{\partial x^w \partial x^s}\delta^u_v + f^s\frac{\partial^2 f^n}{\partial x^s \partial x^v}\delta^u_w + f^n\frac{\partial^2 f^n}{\partial x^w \partial x^n}\delta^u_v + f^n\frac{\partial^2 f^n}{\partial x^n \partial x^v}\delta^u_w +$$

$$\left. + \frac{\partial f^n}{\partial x^n}\frac{\partial f^n}{\partial x^v}\delta^u_w + 2\frac{f^u}{f^n}\frac{\partial f^n}{\partial x^w}\frac{\partial f^n}{\partial x^v} - f^u\frac{\partial^2 f^n}{\partial x^w \partial x^v} + f^n\frac{\partial^2 f^u}{\partial x^w \partial x^v}\right)dx^n. \tag{14}$$

The above formulas show that forms ω_u^n, ω_{uv}^n and ω_{vw}^u are expressed through basic forms ω^u, ω^n and ω^{n+1}. By Theorem 2 this implies that an associated affine connection is adjoined to the system of differential equations (4) in a natural way. Let us call it *a canonical connection* of the system. And the curvature tensor of the connection will be called *the curvature tensor of the system of ordinary differential equations*.

After a number of calculations we find the expression of components of the curvature tensor through functions f^i that define the system (4):

$$R_{vws}^u = -\frac{1}{2\left(f^n\right)^3}\left(\frac{2}{f^n}\frac{\partial f^n}{\partial x^w}\frac{\partial f^n}{\partial x^v} - \frac{\partial^2 f^n}{\partial x^w \partial x^v}\right)\delta_s^u + \frac{1}{2\left(f^n\right)^3}\left(\frac{2}{f^n}\frac{\partial f^n}{\partial x^s}\frac{\partial f^n}{\partial x^v} - \frac{\partial^2 f^n}{\partial x^s \partial x^v}\right)\delta_w^u;$$

$$R_{vwn}^u = -\frac{1}{2\left(f^n\right)^2}\left(-\frac{\partial f^n}{\partial x^w}\frac{\partial f^u}{\partial x^v} - \frac{\partial f^u}{\partial x^w}\frac{\partial f^n}{\partial x^v} - 2\frac{f^s}{f^n}\frac{\partial f^n}{\partial x^w}\frac{\partial f^n}{\partial x^s}\delta_v^u - 2\frac{f^s}{f^n}\frac{\partial f^n}{\partial x^s}\frac{\partial f^n}{\partial x^v}\delta_w^u - \frac{\partial f^n}{\partial x^w}\frac{\partial f^n}{\partial x^n}\delta_v^u - \right.$$

$$- \frac{\partial f^n}{\partial x^n}\frac{\partial f^n}{\partial x^v}\delta_w^u + \frac{\partial f^n}{\partial x^s}\frac{\partial f^s}{\partial x^w}\delta_v^u + \frac{\partial f^n}{\partial x^s}\frac{\partial f^s}{\partial x^v}\delta_w^u + f^s\frac{\partial^2 f^n}{\partial x^w \partial x^s}\delta_v^u + f^s\frac{\partial^2 f^n}{\partial x^s \partial x^v}\delta_w^u + f^n\frac{\partial^2 f^n}{\partial x^w \partial x^n}\delta_v^u +$$

$$\left. + f^n\frac{\partial^2 f^n}{\partial x^n \partial x^v}\delta_w^u + 2\frac{f^u}{f^n}\frac{\partial f^n}{\partial x^w}\frac{\partial f^n}{\partial x^v} - f^u\frac{\partial^2 f^n}{\partial x^w \partial x^v} + f^n\frac{\partial^2 f^u}{\partial x^w \partial x^v}\right);$$

$$R_{vwn+1}^u = \frac{1}{2\left(f^n\right)^2}\left(-\frac{1}{f^n}\frac{\partial f^n}{\partial x^v}\frac{\partial f^n}{\partial t}\delta_w^u - \frac{1}{f^n}\frac{\partial f^n}{\partial x^w}\frac{\partial f^n}{\partial t}\delta_v^u + \frac{\partial^2 f^n}{\partial t \partial x^v}\delta_w^u + \frac{\partial^2 f^n}{\partial x^w \partial t}\delta_v^u\right);$$

$$R_{vnn+1}^u = R_{nvn+1}^u = \frac{1}{2f^n}\left(\frac{\partial f^u}{\partial x^v}\frac{\partial f^n}{\partial t} + f^u\frac{\partial^2 f^n}{\partial x^v \partial t} - \frac{\partial f^n}{\partial x^v}\frac{\partial f^u}{\partial t} - f^n\frac{\partial^2 f^u}{\partial x^v \partial t} + \frac{f^w}{f^n}\frac{\partial f^n}{\partial t}\frac{\partial f^n}{\partial x^w}\delta_v^u - \frac{\partial f^w}{\partial t}\frac{\partial f^n}{\partial x^w}\delta_v^u\right);$$

$$R_{nnn+1}^u = \frac{1}{2}\left(f^u f^v\frac{\partial^2 f^n}{\partial x^v \partial t} - f^v f^n\frac{\partial^2 f^u}{\partial x^v \partial t} + f^u f^n\frac{\partial^2 f^n}{\partial x^n \partial t} - (f^n)^2\frac{\partial^2 f^u}{\partial x^n \partial t} + \right.$$

$$\left. + f^n\frac{\partial f^u}{\partial x^n}\frac{\partial f^n}{\partial t} + f^n\frac{\partial f^v}{\partial t}\frac{\partial f^u}{\partial x^v} - f^u\frac{\partial f^n}{\partial t}\frac{\partial f^n}{\partial x^n} - f^u\frac{\partial f^v}{\partial t}\frac{\partial f^n}{\partial x^v}\right);$$

$$R_{uvn}^n = -\frac{1}{2\left(f^n\right)^3}\left(\frac{\partial^2 f^n}{\partial x^v \partial x^u} - \frac{4}{f^n}\frac{\partial f^n}{\partial x^u}\frac{\partial f^n}{\partial x^v}\right);$$

$$R_{unn+1}^n = R_{nun+1}^n = R_{n+1un+1}^{n+1} = \frac{1}{2\left(f^n\right)^2}\left(\frac{1}{f^n}\frac{\partial f^n}{\partial t}\frac{\partial f^n}{\partial x^u} - \frac{\partial^2 f^n}{\partial x^u \partial t}\right);$$

$$R_{nnn+1}^n = \frac{1}{2f^n}\left(\frac{\partial f^n}{\partial t}\frac{\partial f^n}{\partial x^n} + \frac{\partial f^n}{\partial x^u}\frac{\partial f^u}{\partial t} - f^u\frac{\partial^2 f^n}{\partial x^u \partial t}\right) - \frac{1}{2}\frac{\partial^2 f^n}{\partial x^n \partial t};$$

$$R_{n+1nn+1}^{n+1} = \frac{1}{2f^n}\left(\frac{\partial f^n}{\partial t}\frac{\partial f^n}{\partial x^n} - f^u\frac{\partial^2 f^n}{\partial x^u \partial t}\right) - \frac{1}{2}\frac{\partial^2 f^n}{\partial x^n \partial t} + \frac{f^u}{2\left(f^n\right)^2}\frac{\partial f^n}{\partial t}\frac{\partial f^n}{\partial x^u}.$$

Let us find the form of the system of ordinary differential equations with a zero curvature tensor. To do this we must put all components of the curvature tensor equal to zero. From the equation $R^u_{vws} = 0$, for $u = w \neq s$, we get

$$\frac{\partial^2 f^n}{\partial x^s \partial x^v} = \frac{2}{f^n}\frac{\partial f^n}{\partial x^s}\frac{\partial f^n}{\partial x^v}.$$

From this and the condition $R^n_{uvn} = 0$ it follows that

$$\frac{\partial f^n}{\partial x^v} = 0.$$

Therefore function f^n depends on the variables x^n and t only, i.e.,

$$f^n = f^n(x^n, t). \tag{15}$$

By (15) in the system $R^a_{bcd} = 0$ only the following equations are left:

$$\frac{\partial^2 f^u}{\partial x^w \partial x^v} = 0; \tag{16}$$

$$\frac{\partial^2 f^u}{\partial x^v \partial t} = \frac{1}{f^n}\frac{\partial f^u}{\partial x^v}\frac{\partial f^n}{\partial t}; \tag{17}$$

$$\frac{\partial^2 f^n}{\partial x^n \partial t} = \frac{1}{f^n}\frac{\partial f^n}{\partial t}\frac{\partial f^n}{\partial x^n}; \tag{18}$$

$$f^v f^n \frac{\partial^2 f^u}{\partial x^v \partial t} - f^u f^n \frac{\partial^2 f^n}{\partial x^n \partial t} + (f^n)^2 \frac{\partial^2 f^u}{\partial x^n \partial t} - f^n \frac{\partial f^u}{\partial x^n}\frac{\partial f^n}{\partial t} - f^n \frac{\partial f^v}{\partial t}\frac{\partial f^u}{\partial x^v} + f^u \frac{\partial f^n}{\partial t}\frac{\partial f^n}{\partial x^n} = 0. \tag{19}$$

Putting (17) and (18) in (19) we have

$$-\frac{f^v}{f^n}\frac{\partial f^u}{\partial x^v}\frac{\partial f^n}{\partial t} - f^n \frac{\partial^2 f^u}{\partial x^n \partial t} + \frac{\partial f^u}{\partial x^n}\frac{\partial f^n}{\partial t} + \frac{\partial f^v}{\partial t}\frac{\partial f^u}{\partial x^v} = 0. \tag{20}$$

Now we integrate the equation (16)

$$f^u = a^u_v(x^n, t)x^v + b^u(x^n, t). \tag{21}$$

Let us write (17) in the form

$$\frac{\partial}{\partial t}\left(\frac{\frac{\partial f^u}{\partial x^v}}{f^n}\right) = 0,$$

or, using (21),

$$\frac{\partial}{\partial t}\left(\frac{a^u_v(x^n, t)}{f^n(x^n, t)}\right) = 0.$$

Hence

$$a^u_v(x^n, t) = c^u_v(x^n) f^n(x^n, t). \tag{22}$$

Now we find the form $f^n(x^n, t)$. Let us assume $f^n = e^{\varphi}$, $\varphi = \varphi(x^n, t)$. Then (18) becomes

$$\varphi_{tn} = 0,$$

whence

$$\varphi = \tilde{q}(x^n) + \tilde{g}(t).$$

We come back to f^n

$$f^n = e^{\tilde{q}(x^n)+\tilde{g}(t)} = q(x^n)g(t). \tag{23}$$

From (22) and (23) it follows that the function f^u (see (21)) will take the form

$$f^u = c^u_v(x^n)q(x^n)g(t)x^v + b^u(x^n,t). \tag{24}$$

We put (23) and (24) in the equation (20). To do this we first find

$$\frac{\partial f^u}{\partial x^n} = \frac{\partial c^u_v(x^n)}{\partial x^n}q(x^n)g(t)x^v + \frac{\partial q(x^n)}{\partial x^n}c^u_v(x^n)g(t)x^v + \frac{\partial b^u(x^n,t)}{\partial x^n};$$

$$\frac{\partial f^u}{\partial t} = c^u_v(x^n)x^v q(x^n)\frac{\partial g(t)}{\partial t} + \frac{\partial b^u(x^n,t)}{\partial t};$$

$$\frac{\partial^2 f^u}{\partial x^n \partial t} = \frac{\partial c^u_v(x^n)}{\partial x^n}\frac{\partial g(t)}{\partial t}q(x^n)x^v + \frac{\partial q(x^n)}{\partial x^n}c^u_v(x^n)\frac{\partial g(t)}{\partial t}x^v + \frac{\partial^2 b^u(x^n,t)}{\partial x^n \partial t}.$$

Substituting this in (20), after some transformations we come to the equation

$$\left(b^v(x^n,t)c^u_v(x^n) - \frac{\partial b^u(x^n,t)}{\partial x^n}\right)\frac{\partial g(t)}{\partial t} + \left(\frac{\partial^2 b^u(x^n,t)}{\partial x^n \partial t} - \frac{\partial b^v(x^n,t)}{\partial t}c^u_v(x^n)\right)g(t) = 0$$

or

$$\frac{\partial}{\partial t}\left(\frac{\frac{\partial b^u(x^n,t)}{\partial x^n} - b^v(x^n,t)c^u_v(x^n)}{g(t)}\right) = 0. \tag{25}$$

Integrating (25) we find

$$\frac{\partial b^u(x^n,t)}{\partial x^n} = b^v(x^n,t)c^u_v(x^n) + g(t)r^u(x^n), \tag{26}$$

where $r^u(x^n)$ are some new functions. Thus the following Theorem is proved.

Theorem 4. The system of ordinary differential equations (4) with zero curvature tensor has the form:

$$\begin{aligned} \frac{dx^u}{dt} &= c^u_v(x^n)q(x^n)g(t)x^v + b^u(x^n,t); \\ \frac{dx^n}{dt} &= q(x^n)g(t). \end{aligned} \tag{27}$$

Here functions $b^u(x^n,t)$ satisfy the equation (26).

This result can be simplified. By admissible substitution of the variables $x^n = x^n(\tilde{x}^n)$ and $t = t(\tilde{t})$ the last equation of (27) takes the form

$$\frac{dx^n}{dt} = 1,$$

i.e., $f^n(x^n,t) = 1$. By integration we get $x^n = t + c$. In this case system (27) takes the form

$$\frac{dx^u}{dt} = C^u_v(t)x^v + B^u(t),$$

where, taking (26) into account, functions $B^u(t)$ satisfy the equations

$$\frac{dB^u}{dt} = B^v(t)C_v^u(t) + R^u(t)$$

and

$$R^u(t) = r^u(t+c) + \frac{\partial b^u(t+c,t)}{\partial t}.$$

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АФФИННАЯ СВЯЗНОСТЬ, ПРИСОЕДИНЕННАЯ К ТКАНИ W(1, N, 1)

Дуюнова А.А.

В статье к три-ткани W(1,n,1) и системе дифференциальных уравнений присоединяется совместимая аффинная связность без кручения, названая канонической связностью системы ОДУ. Компоненты тензора кривизны этой связности вычислены через функции, определяющие систему дифференциальных уравнений, и записан вид системы ОДУ с нулевым тензором кривизны.

Ключевые слова: три-ткань, система обыкновенных дифференциальных уравнений, аффинная связность.

About the author

Duyunova Anna Andreevna, graduated Saratov State University named after Chernyshevskiy (2007), Ph.D. in physical and mathematical sciences, author of 6 published scientific papers.