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ON THE STRUCTURE OF THE OPERATOR COADJOINT ACTION FOR THE CURRENT ALGEBRA ON THE THREE-DIMENSIONAL TORUS

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For the current Lie algebra on the three-dimensional torus with non-standard Lie bracket some properties, in the case when the sum of adjoint and coadjoint operators on infinite-dimensional Lie algebra with scalar product has a finite norm are established. For the Landau-Lifshitz equation in the three-dimensional torus it is established that the operator $\mathrm{S}_m = (\mathrm{ad}_m + \mathrm{ad}^*_m)/2$ has a finite norm, though it is not true the operators of the adjoint action ad_m and coadjoint action ad^*_m. It follows that the coefficients of expansion of the solution in an orthonormal basis of eigenvectors of the Laplace operator satisfy Lipschitz conditions. Thus, for the Landau-Lifshitz equation on the three-dimensional torus situation is similar to the equations of an ideal fluid and Korteweg de Vries. On the other hand, if for the equations of fluid dynamics and Korteweg de Vries, this fact has been established in a general way, for the Landau-Lifshitz equation in the three-dimensional torus it is obtained specifically through the calculation of structural constants and the matrix of the coadjoint action for the current algebra with non-standard Lie bracket.

Key words: current algebra, Lie bracket, operator of the adjoint action, operator of the coadjoint action, three-torus, the Landau-Lifshitz equation, compact operator, Lipschitz condition.

INTRODUCTION

Let us denote the three-dimensional torus by T^3. Denote by $V(\mathrm{T}^3)$ the space of smooth vector fields on T^3. In the space $V(\mathrm{T}^3)$ there exists the standard current algebra structure $g(\mathrm{T}^3, so(3))$, i.e. the Lie algebra of vector fields on T^3 with the operation of the pointwise vector product:

$$(m\times n)(x) = m(x)\times n(x). \qquad (1)$$

Consider the Landau-Lifshitz equation on T^3:

$$\frac{\partial m}{\partial t} = m\times \Delta m, \qquad (2)$$

Here $m\in V(T^3)$, Δ – Laplace operator on vector fields.

We can represent the Landau-Lifshitz equation as Euler equation in the current algebra but with a non-standard Lie bracket [1–4] (see also [6–7] for the invariant approach methods used). For this we fix $a>0$ and introduce the operator:

$$P_a = -a\mathrm{Id} + \Delta,$$

where Id is the identity operator. The operator P_a is invertible.

Let there be given in the space $V(\mathrm{T}^3)$ a non-standard Lie bracket:

$$\mathrm{ad}_m n = [m,n] = P_a^{-1}(P_a m \times P_a n). \qquad (3)$$

Define also the scalar product in $V\left(\mathrm{T}^3\right)$:

$$< m, n > = \int_{T^3} (m(x), -P_a(n)(x))\, dx . \tag{4}$$

Denote by $\mathrm{ad}^*_{\mathrm{m}} n$ the operator of coadjoint action in the sense of metric (4):

$$< \mathrm{ad}^*_m (u), v > = < u, \mathrm{ad}_{\mathrm{m}} (v) > . \tag{5}$$

According to [4] the operator of coadjoint action has the form:

$$\mathrm{ad}^*_m (n) = -P_a \mathrm{ad}_m P_a^{-1} (n) = -P_a m \times n . \tag{6}$$

Then the Landau-Lifshitz equation has the form of the Euler equation [4] on the current algebra with Lie bracket (3) and scalar product (4).

$$\frac{\partial m}{\partial t} = \mathrm{ad}^*_m (m) . \tag{7}$$

Introduce the operator

$$\mathrm{S}_m = \frac{1}{2} (\mathrm{ad}_m + \mathrm{ad}^*_m) . \tag{8}$$

Construct the orthonormal basis of the Lie algebra $V(\mathrm{T}^3)$ in the sense of the metric (4) using eigenvectors of the operator P_a:

$$e_k^i = \frac{\lambda_k}{\sqrt{(|k|^2 + a)}} (\delta_{i,1}, \delta_{i,2}, \delta_{i,3}) \cos(kx),\; f_k^i = \frac{\lambda_k}{\sqrt{(|k|^2 + a)}} (\delta_{i,1}, \delta_{i,2}, \delta_{i,3}) \sin(kx), k \neq 0. \tag{9}$$

Here $k \in \mathrm{Z}^3$, $i = 1, 2, 3$, $\lambda = \sqrt{\dfrac{2}{vol\left(\mathrm{T}^3\right)}}$, $k \neq 0$; $\lambda_0 = \sqrt{\dfrac{1}{vol\left(\mathrm{T}^3\right)}}$, $\delta_{i,j} = 1$ if $i = j$, otherwise 0.

For uniqueness it is convenient to represent an integer vector $k \neq 0$ (that is, the index of an element of basis (9)) in the form $k = \varepsilon(k)\bar{\bar{k}}$, where the first non-zero coordinate of the vector $\bar{\bar{k}}$ is positive and $\varepsilon(k) \in \{1, -1\}$. We use notation $[\mathrm{Z}^3] = \{\bar{\bar{k}} \mid k \in \mathrm{Z}^3 \setminus \{0\}\} = \mathrm{Z}^3 \setminus \{0\} / \{\pm 1\}$.

Let $m(t)$ be a solution of (7). Expand it to the orthonormal basis (9):

$$m(t) = \sum_i g_{0,i}(t) e_0^i + \sum_{k \in [Z^3], i} (g_{k,i}(t) e_k^i + h_{k,i}(t) f_k^i) . \tag{10}$$

According to [4], the Landau-Lifshitz equation implies the following equation for the coefficients of expansion (10):

$$\frac{\partial g_{k,i}}{\partial t} = - < \mathrm{S}_{e_k^i}\, m, m >, \frac{\partial h_{k,i}}{\partial t} = - < \mathrm{S}_{f_k^i}\, m, m > . \tag{11}$$

In [5] a class of infinite-dimensional Lie with scalar product has been introduced, for which the operator S_m has a finite norm. This property has been established for the Lie algebra of divergence-free vector fields on a compact Riemannian manifold (the configuration space of an ideal incompressible fluid); Lie algebra of vector fields on a compact Riemannian manifold (the configuration space of an ideal compressible fluid); Virasoro algebra (the configuration space of the Korteweg-de Vries equation). It follows that for the Euler solutions' expansion coefficients in the orthonormal basis of the Lie algebra satisfy Lipschitz conditions.

In this paper, this feature is established for the Landau-Lifshitz equation in the three-torus.

1. THE STRUCTURE PROPERTIES OF THE CURRENT ALGEBRA $V(\mathrm{T}^3)$ WITH A NON-STANDARD LIE BRACKET

To calculate the structure constants of Lie algebra $V(\mathrm{T}^3)$ we fix $i \neq j$. Denote $\delta_i = (\delta_{i,1}, \delta_{i,2}, \delta_{i,3})$. It can be assumed that without loss of generality $\delta_i \times \delta_j = \delta_s$. It is convenient to introduce:

$$\alpha_{k,l} = \frac{\sqrt{a+|k+l|^2}}{\sqrt{a+|l|^2}}, \quad \beta_{k,l} = \frac{\sqrt{a+|k-l|^2}}{\sqrt{a+|l|^2}}. \tag{12}$$

Using (3), (6) we see

$$\mathrm{ad}_{e_k^i} e_l^j = \lambda_k \lambda_l \sqrt{(a+|k|^2)(a+|l|^2)}\, P_a^{-1}(\cos kx \cos lx)\,\delta_s =$$

$$= -\lambda_k \lambda_l \sqrt{(a+|k|^2)(a+|l|^2)}\, \frac{1}{2}\left(\frac{\cos(k+l)}{a+|(k+l)|^2}\delta^s + \frac{\cos(k-l)}{a+|(k-l)|^2}\delta^s \right) =$$

$$= -\frac{\lambda_k}{2}\sqrt{(a+|k|^2)}\left(\frac{\lambda_l}{\lambda_{k+l}\alpha_{k,l}} e_{k+l}^s + \frac{\lambda_l}{\lambda_{k-l}\beta_{k,l}} e_{\overline{\overline{k-l}}}^s \right);$$

$$\mathrm{ad}^*_{e_k^i} e_l^j = \lambda_k \lambda_l \sqrt{\frac{a+|k|^2}{a+|l|^2}} \cos kx \cos lx \delta_s =$$

$$= \frac{\lambda_k}{2}\sqrt{a+|k|^2}\left(\frac{\lambda_l \sqrt{a+|k+l|^2}\, e_{k+l}}{\lambda_{k+l}\sqrt{a+|l|^2}} + \frac{\lambda_1 \sqrt{a+|k-l|^2}\, e_{\overline{\overline{k-l}}}}{\lambda_{k-l}\sqrt{a+|l|^2}} \right) = \tag{13}$$

$$= \frac{\lambda_k}{2}\sqrt{\left(a+|k|^2\right)}\left(\frac{\lambda_l \alpha_{k,l}}{\lambda_{k+l}} e_{k+l}^s + \frac{\lambda_l \beta_{k,l}}{\lambda_{k-l}} e_{\overline{\overline{k-l}}}^s \right).$$

Further, we have

$$\mathrm{ad}_{e_k^i} f_l^j = \lambda_k \lambda_l \sqrt{(a+|k|^2)(a+|l|^2)}\, P_a^{-1}(\cos kx \sin lx)\,\delta_s =$$

$$= \frac{\lambda_k}{2}\sqrt{(a+|k|^2)}\left(-\frac{\lambda_l}{\lambda_{k+l}\alpha_{k,l}} f_{k+l}^s + \frac{\lambda_l}{\lambda_{k-l}\beta_{k,l}} \varepsilon(k-l) f_{\overline{\overline{k-l}}}^s \right), \text{ if } k \neq l,$$

$$\mathrm{ad}_{e_k^i} f_k^j = -\frac{\lambda_k^2}{2}\sqrt{(a+|k|^2)}\, \frac{1}{\lambda_{2k}\alpha_{k,k}} f_{2k}^s;$$

$$\begin{aligned}
&\mathrm{ad}^*_{e_k^i} f_l^j = \lambda_k \lambda_l \sqrt{\frac{a+|k|^2}{a+|l|^2}} \cos kx \sin lx\, \delta_{\mathrm{s}} = \\
&= \frac{\lambda_k}{2}\sqrt{(a+|k|^2)}\left(\frac{\lambda_l \alpha_{k,l}}{\lambda_{k+l}} f^s_{k+l} - \frac{\lambda_l \beta_{k,l}}{\lambda_{k-l}} \varepsilon(k-l)\, f^s_{\overline{\overline{k-l}}}\right), \text{ if } k \neq l, \\
&\mathrm{ad}^*_{e_k^s} f_k^s = \frac{\lambda_k^2}{2}\sqrt{(a+|k|^2)}\,\frac{\alpha_{k,k}}{\lambda_{2k}} f^s_{2k}.
\end{aligned} \tag{14}$$

We have also

$$\begin{aligned}
&\mathrm{ad}_{f_k^i} f_l^j = \lambda_k \lambda_l \sqrt{(a+|k|^2)(a+|l|^2)}\, P_a^{-1}(\sin kx \sin lx)\, \delta_{\mathrm{s}} = \\
&\frac{\lambda_k}{2}\sqrt{(a+|k|^2)}\left(\frac{\lambda_l}{\lambda_{k+l}\alpha_{k,l}} e^s_{k+l} - \frac{\lambda_l}{\lambda_{k-l}\beta_{k,l}} e^s_{\overline{\overline{k-l}}}\right);
\end{aligned}$$

$$\begin{aligned}
&\mathrm{ad}^*_{f_k^i} f_l^j = \lambda_k \lambda_l \sqrt{\frac{a+|k|^2}{a+|l|^2}} \sin kx \sin lx\, \delta_{\mathrm{s}} = \\
&= \frac{\lambda_k}{2}\sqrt{(a+|k|^2)}\left(-\frac{\lambda_l \alpha_{k,l}}{\lambda_{k+l}} e^s_{k+l} + \frac{\lambda_l \beta_{k,l}}{\lambda_{k-l}} e^s_{\overline{\overline{k-l}}}\right).
\end{aligned} \tag{15}$$

And further,

$$\begin{aligned}
&\mathrm{ad}_{f_k^i} e_l^j = -\,\mathrm{ad}_{e_l^j} f_k^i = \\
&\frac{\lambda_k}{2}\sqrt{(a+|k|^2)}\left(-\frac{\lambda_l}{\lambda_{k+l}\alpha_{k,l}} f^s_{k+l} - \frac{\lambda_l}{\lambda_{k-l}\beta_{k,l}} \varepsilon(k-l)\, f^s_{\overline{\overline{k-l}}}\right), \text{ if } k \neq l, \\
&\mathrm{ad}_{f_k^i} e_k^j = -\frac{\lambda_k^2}{2}\sqrt{(a+|k|^2)}\,\frac{1}{\lambda_{2k}\alpha_{k,k}} f^s_{2k}; \\
&\mathrm{ad}^*_{f_k^i} e_l^j = \lambda_k \lambda_l \sqrt{\frac{a+|k|^2}{a+|l|^2}} \sin kx \cos lx\, \delta_{\mathrm{s}} = \\
&= \frac{\lambda_k}{2}\sqrt{(a+|k|^2)}\left(\frac{\lambda_l \alpha_{k,l}}{\lambda_{k+l}} f^s_{k+l} + \frac{\lambda_l \beta_{k,l}}{\lambda_{k-l}} \varepsilon(k-l)\, f^s_{\overline{\overline{k-l}}}\right), \text{ if } k \neq l, \\
&\mathrm{ad}^*_{f_k^i} f_l^j = \frac{\lambda_k^2}{2}\sqrt{(a+|k|^2)}\,\frac{\alpha_{k,k}}{\lambda_{2k}} f^s_{2k}.
\end{aligned} \tag{16}$$

Remark 1. From the method of indexing elements of the basis (9) specified above, codes $k+l$ in the formulas (13)–(16) for different vectors l provide different elements of $[\mathrm{Z}^3]$, and therefore, they correspond to different elements of the basis (9). To index $\overline{\overline{k-l}}$ one element $s \in [\mathrm{Z}^3]$ may in certain cases correspond to exactly two different vectors l, and, consequently, to two different elements of the basis (9). This is possible if $\overline{\overline{k-l}}_1 = k-l_1 = s \in [\mathrm{Z}^3]$; in this case denote $l_1 = k-s$. Then for $l_2 = k+s \neq l_1$ we have $\overline{\overline{k-l_2}} = \overline{\overline{-s}} = s$.

Denote

$$\mu_{k,l} = \alpha_{k,l} - \frac{1}{\alpha_{k,l}}, \ \nu_{k,l} = \beta_{k,l} - \frac{1}{\beta_{k,l}}. \tag{17}$$

Using the formulas (13)–(16) we see that the operator (8) has the form:

$$\begin{aligned}
&\mathrm{S}_{e_k^i} e_l^j = \frac{\lambda_k}{4}\sqrt{(a+|k|^2)}\left(\frac{\lambda_l}{\lambda_{k+l}}\mu_{k,l}\, e_{k+l}^s + \frac{\lambda_l}{\lambda_{k-l}}\nu_{k,l} e_{\overline{\overline{k-l}}}^s\right);\\
&S_{e_k^i} f_l^j = \frac{\lambda_k}{4}\sqrt{\left(a+|k|^2\right)}\left(\frac{\lambda_l}{\lambda_{k+l}}\mu_{k,l}\, f_{k+l}^s - \frac{\lambda_l}{\lambda_{k-l}}\nu_{k,l}\, f_{\overline{\overline{k-l}}}^s\right), \text{if } k \neq l;\\
&S_{e_k^i} f_k^j = \frac{\lambda_k^2}{4}\sqrt{\left(a+|k|^2\right)}\frac{1}{\lambda_{2k}}\mu_{k,k}\, f_{2k}^s.
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\mathrm{S}_{f_k^i} f_l^j = \frac{\lambda_k}{4}\sqrt{(a+|k|^2)}\left(-\frac{\lambda_l}{\lambda_{k+l}}\mu_{k,l}\, e_{k+l}^s + \frac{\lambda_l}{\lambda_{k-l}}\nu_{k,l} e_{\overline{\overline{k-l}}}^s\right);\\
&S_{f_k^i} e_l^j = \frac{\lambda_k}{4}\sqrt{\left(a+|k|^2\right)}\left(\frac{\lambda_l}{\lambda_{k+l}}\mu_{k,l}\, f_{k+l}^s + \frac{\lambda_l}{\lambda_{k-l}}\nu_{k,l}\, f_{\overline{\overline{k-l}}}^s\right), \text{if } k \neq l;\\
&S_{f_k^i} e_k^j = \frac{\lambda_k^2}{4}\sqrt{\left(a+|k|^2\right)}\frac{1}{\lambda_{2k}}\mu_{k,k}\, f_{2k}^s.
\end{aligned} \tag{19}$$

Then we transform the expressions for $\mu_{k,l}, \nu_{k,l}$.

$$\begin{aligned}
\mu_{k,l} &= \alpha_{\kappa,l} - \frac{1}{\alpha_{\kappa,l}} = \frac{2(k,l)+|k|^2}{\sqrt{(a+|k+l|^2)(a+|l|^2)}},\\
\nu_{k,l} &= \beta_{\kappa,l} - \frac{1}{\beta_{\kappa,l}} = \frac{-2(k,l)+|k|^2}{\sqrt{(a+|k-l|^2)(a+|l|^2)}}.
\end{aligned} \tag{20}$$

We obtain an asymptotic form of the formulas (20) at $|l| \to \infty$.

$$\frac{1}{\sqrt{\left(a+|k+l|^2\right)\left(a+|l|^2\right)}} = \frac{1}{|l|^2\sqrt{\left(1+\frac{2(k,l)}{|l|^2}+\frac{|k|^2}{|l|^2}+\frac{a}{|l|^2}\right)\left(1+\frac{a}{|l|^2}\right)}} = \frac{1}{|l|^2} - \frac{(k,l)}{|l|^4} + o\left(\frac{1}{|l|^3}\right),$$

$$\frac{1}{\sqrt{\left(a+|k-l|^2\right)\left(a+|l|^2\right)}} = \frac{1}{|l|^2\sqrt{\left(1-\frac{2(k,l)}{|l|^2}+\frac{|k|^2}{|l|^2}+\frac{a}{|l|^2}\right)\left(1+\frac{a}{|l|^2}\right)}} = \frac{1}{|l|^2} + \frac{(k,l)}{|l|^4} + o\left(\frac{1}{|l|^3}\right).$$

Hence

$$\mu_{k,l}=(2(k,l)+|k|^2)(\frac{1}{|l|^2}-\frac{(k,l)}{|l|^4}+o(\frac{1}{|l|^3}))=\frac{2(k,l)}{|l|^2}+\frac{|k|^2}{|l|^2}-\frac{2(k,l)^2}{|l|^4}+o(\frac{1}{|l|^2}),$$
$$\nu_{k,l}=(-2(k,l)+|k|^2)(\frac{1}{|l|^2}+\frac{(k,l)}{|l|^4}+o(\frac{1}{|l|^3}))=-\frac{2(k,l)}{|l|^2}+\frac{|k|^2}{|l|^2}-\frac{2(k,l)^2}{|l|^4}+o(\frac{1}{|l|^2}). \quad (21)$$

Note that for $|k|<|l|$ we have $\lambda_l=\lambda_{k+l}=\lambda_{k-l}$, and also $\mu_{0,l}=\nu_{0,l}=0$. Therefore, we consider the case $k\neq 0$ for investigation of the asymptotic behavior (20). Denote

$$\lambda_k=\sqrt{\frac{2}{vol(T^3)}}=\lambda,\ a_{k,l}=\frac{2(k,l)}{|l|^2},\ b_{k,l}=\frac{|k|^2}{|l|^2}-\frac{2(k,l)^2}{|l|^4}.$$

We obtain the following asymptotic formulas:

$$\mathrm{S}_{e_k^i}e_l^j=\frac{\lambda}{4}\sqrt{(a+|k|^2)}\left((a_{k,l}+b_{k,l}+o(\frac{1}{|l|^2}))e_{k+l}^s+(-a_{k,l}+b_{k,l}+o(\frac{1}{|l|^2}))e_{\overline{\overline{k-l}}}^s\right);$$
$$\mathrm{S}_{e_k^i}f_l^j=\frac{\lambda}{4}\sqrt{(a+|k|^2)}\left((a_{k,l}+b_{k,l}+o(\frac{1}{|l|^2}))f_{k+l}^s+(a_{k,l}-b_{k,l}+o(\frac{1}{|l|^2}))f_{\overline{\overline{k-l}}}^s\right);\quad (22)$$
$$\mathrm{S}_{f_k^i}e_l^j=\frac{\lambda}{4}\sqrt{(a+|k|^2)}\left((a_{k,l}+b_{k,l}+o(\frac{1}{|l|^2}))f_{k+l}^s+(-a_{k,l}+b_{k,l}+o(\frac{1}{|l|^2}))f_{\overline{\overline{k-l}}}^s\right);$$
$$\mathrm{S}_{f_k^i}f_l^j=\frac{\lambda}{4}\sqrt{(a+|k|^2)}\left((-a_{k,l}-b_{k,l}+o(\frac{1}{|l|^2}))e_{k+l}^s+(-a_{k,l}+b_{k,l}+o(\frac{1}{|l|^2}))e_{\overline{\overline{k-l}}}^s\right).$$

Proposision 1. The operators $\mathrm{S}_{e_k^i},\mathrm{S}_{f_k^i}$ are compact in the sense of the L^2-metric (4).

This follows from the formulas (21)–(22), Remark 1 and [6, p. 234].

Proposition 2. The norms of operators $\mathrm{S}_{e_k^i},\mathrm{S}_{f_k^i}$ in the sense of the L^2-metric (4) are evaluated as follows

$$\left\|\mathrm{S}_{e_k^i}\right\|,\left\|\mathrm{S}_{f_k^i}\right\|\le\lambda(\sqrt{2}+2)|k|)\sqrt{\frac{(a+|k|^2)}{a}}\ . \quad (23)$$

To prove this, we estimate the components of (20).

$$\left|\frac{2(k,l)}{\sqrt{(a+|k+l|^2)(a+|l|^2)}}\right|\le\frac{2\left|(k,\frac{l}{|l|})\right|}{\sqrt{(a+|k+l|^2)}}\le 2\frac{|k|}{\sqrt{a}}\ .$$

Note that $|k+l|^2+|l|^2=|k|^2+2|l|^2+2(k,l)\geq|k|^2+2|l|^2-2|k||l|=2(|l|-\frac{|k|}{2})^2+\frac{|k|^2}{2}\geq\frac{|k|^2}{2}$.

Consequently, we have $\frac{|k|^2}{\sqrt{\left(a+|k+l|^2\right)\left(a+|l|^2\right)}}\leq\frac{|k|^2}{\sqrt{a\frac{|k|^2}{2}}}\leq|k|\sqrt{\frac{2}{a}}$. Then we obtain the estimations $|\mu_{k,l}|,\ |\nu_{k,l}|\leq(2+\sqrt{2})\frac{|k|}{\sqrt{a}}$.

In conclusion, we use Remark 1 and the formulas (18), (19).

Corollary 1. The coefficients $g_{k,i}(t),\ h_{k,i}(t)$ of the expansion (10) in the orthonormal basis (9) of a solution $m(t)$ of Landau-Lifshitz equation satisfy Lipschitz condition with respect to t with the Lipschitz constant:

$$\lambda(\sqrt{2}+2)|k|)\sqrt{\frac{(a+|k|^2)}{a}}\|m(0)\|^2 . \tag{24}$$

Theorem 1. For any element $m\in V(T^3)$ the operator S_m has a finite norm in the sense of L^2-metric (4).

Proof. Represent m in the form (10).

$$m=\sum_i g_{0,i}\,e_0^i+\sum_{k\in[Z^3],i}(g_{k,i}\,e_k^i+h_{k,i}\,f_k^i). \tag{25}$$

Since the vector field m has class C^∞, then from the expansion properties of the Fourier series [7] and the fact that elements of (9) differ from element of standard Fourier-basis multiplier of the order $\frac{1}{|k|}$, we have that for any natural n there exist such $C_n>0, N_n>0$ that the following estimations hold:

$$|g_{k,i}|,|h_{k,i}|\leq\frac{C_n}{|k|^n},k\in[\mathrm{Z}^3],|k|\geq N_n . \tag{26}$$

It follows that

$$\left\|S_{g_{k,i}\,e_k^i}\right\|,\ \left\|S_{h_{k,i}\,f_k^i}\right\|\leq\frac{C_n}{|k|^n}\lambda(\sqrt{2}+2)|k|)\sqrt{\frac{(a+|k|^2)}{a}} .$$

Using (26) for $n\geq 6$ we see that the series

$$S_m=\sum_i S_{g_{0,i}\,e_0^i}+\sum_{k\in[Z^3],i}(S_{g_{k,i}\,e_k^i}+S_{h_{k,i}\,f_k^i})$$

converges in the norm of the linear operator.

CONCLUSION

For the Landau-Lifshitz equation in the three-dimensional torus it is established that the operator S_m (8) has a finite norm, though it is not true the operators of the adjoint action ad_m and coadjoint action ad_m^* of the final regulations. It follows that the coefficients of expansion of the solution in an orthonormal basis of eigenvectors of the Laplace operator satisfy Lipschitz conditions. Thus, for the Landau-Lifshitz equation on the three-dimensional torus situation is similar to the equations of an ideal fluid and Korteweg de Vries. On the other hand, if for the equations of fluid dynamics and Korteweg de Vries, this fact has been established in a general way, for the Landau-Lifshitz equation in the three-dimensional torus it is obtained specifically through the calculation of structural constants and the matrix of the coadjoint action for the current algebra with non-standard Lie bracket.

REFERENCES

1. Arnold V.I., Khesin B.A. Topological methods in gydrodynamics. Springer, New-York, 1998, 392 p.

2. Khesin B.A., Wendt R. The geometry of infinite-dimensional group. Springer. New-York, 2009, 304 p.

3. Aleksovskii V.A., Lukatskii A.M. Nonlinear dynamics of the magnetization of ferromagnets and motion of a generalized solid with flow group. Theoret. and Math. Phys., 1990, vol. 85, no. 1, pp. 1090–1096.

4. Lukatsky A.M. Structural and geometric properties infinite Lie groups in the application of the equations of mathematical physics. Yaroslavl, Yaroslavl State University named P.G. Demidov, 2010, 175 p. (in Russian)

5. Lukatsky A.M. Investigation of the geodesic flow on an infinite-dimensional Lie group by means of the coadjoint action operator. Proceedings of the Steklov Institute of Mathematics, December 2009, vol. 267, issue 1, pp. 195–204. (in Russian)

6. Kolmogorov A.N., Fomin S.V. Elements of the theory of functions and functional analysis. M., Nauka, 1972, 496 p. (in Russian)

7. Zhuk V.V., Natanson G.I. Trigonometric Fourier series and elements approximation theory. Leningrad. Publishing House of Leningrad University Press, 1983, 188 p. (in Russian)

8. Temam R. Navier-Stokes equations. Theory and numerical analysis. North Holland Publ. Comp., 1979.

9. Arnold V.I. Mathematical methods in classical mechanics. M., Editorial URSS, 2000, 408 p. (in Russian)

10. Zulanke R., Witten P. Differential geometry and fiber bundles. M., Mir, 1975. (in Russian)

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О СТРУКТУРЕ ДЕЙСТВИЯ КОПРИСОЕДИНЕННОГО ОПЕРАТОРА НА АЛГЕБРЕ ТОКОВ ТРЕХМЕРНОГО ТОРА

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Для алгебры Ли потоков на трехмерном торе с нестандартной скобкой Ли установлены некоторые свойства, в случае когда сумма присоединённого и коприсоединенного операторов на бесконечномерной алгебре Ли со скалярным произведением, имеет конечную норму. Точнее, для уравнения Ландау – Лифшица на трехмерном торе установлено, что оператор $S_m = (ad_m + ad_m^*) / 2$ имеет конечную норму, хотя это не так для присоединённого действия ad_m и коприсоединённого действия ad_m^*. Из этого выводится, что коэффициенты разложения решения по ортонормированному базису собственных векторов оператора Лапласа удовлетворяют условию Липшица. Таким образом, для уравнения Ландау – Лифшица на трехмерном торе ситуация схожа с таковой для идеальной жидкости и уравнения Кортвега – де Фриза. С другой стороны, если для уравнений гидродинамики и уравнения Кортвега – де Фриза такой факт был установлен в общем виде, то для уравнения Ландау – Лифшица на трехмерном торе это выведено специальным способом, через вычисление структурных констант и матрицы коприсоединённого действия на алгебре токов с нестандартной скобкой Ли.

Ключевые слова: алгебра токов, скобка Ли, действие присоединённого оператора, оператор коприсоединённого действия, трёхмерный тор, уравнение Ландау – Лифшица, компактный оператор, условие Липшица.

СПИСОК ЛИТЕРАТУРЫ

1. Arnold V.I., Khesin B.A. Topological methods in gydrodynamics. Springer, New-York, 1998, 392 p.

2. Khesin B.A., Wendt R. The geometry of infinite-dimensional group. Springer. New-York, 2009, 304 p.

3. Aleksovskii V.A., Lukatskii A.M. Nonlinear dynamics of the magnetization of ferromagnets and motion of a generalized solid with flow group. Theoret. and Math. Phys., 1990, vol. 85, no. 1, pp. 1090–1096.

4. Лукацкий А.М. Структурные и геометрические свойства бесконечномерных групп Ли в приложении к уравнениям математической физики. Ярославль: Ярославский Государственный Университет, 2010. 175 с.

5. Лукацкий А.М. Исследование геодезических потоков на бесконечномерных группах Ли при помощи действия коприсоединенного оператора // Труды Математического института им. Стеклова. 2009. Т. 267, вып. 1. С. 195–204.

6. Колмогоров А.Н., Фомин С.В. Элементы теории функций и функционального анализа. М.: Наука, 1972. 496 с.

7. Жук В.В., Натанзон Г.И. Тригонометрические ряды Фурье и элементы теории аппроксимаций. Л.: Издательство Ленинградского Университета, 1983. 188 с.

8. Temam R. Navier-Stokes equations. Theory and numerical analysis. North Holland Publ. Comp., 1979.

9. Арнольд В.И. Математические методы в классической механике. М.: Эдиториал УРСС, 2000. 408 с.

10. Зуланке Р., Виттен П. Дифференциальная геометрия и расслоения. М.: Мир, 1975.

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